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AN

ELEMENTARY TREATISE

ON

ANALYTIC GEOMETRY,

EMBRACING

PLANE GEOMETRY

AND AN

INTRODUCTION TO GEOMETRY OF THREE DIMENSIONS.

BY

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PREFACE.

THE present work on Analytic Geometry is designed as a text-book for Colleges and Scientific Schools. The object has been to exhibit the subject in a clear and simple manner, especially for the use of beginners, and at the same time to include all that students usually require in the regular undergraduate course.

The aim has been to make every principle clear, so as to be easily understood by students of average ability; for I venture to think that there is nothing in the subject that lies beyond the reach of good common minds.

It has been the intention to furnish a useful text-book; clear and concise demonstrations have been sought and freely taken from all available sources. Wherever it has been possible to simplify a line of proof as indicated by the leading writers, so as to adapt it to the use of beginners, it has been done; but no attempt has been made to depart from clear and satisfactory methods adopted by others, merely for the purpose of seeming to be original.

It is thought that among the merits of this book are the presentation of the symmetrical and normal forms of the equations of the right line and of the plane, the equations of the ellipsoid and of the plane tangent to the ellipsoid, and the formulæ for the distances of a point from a line and from a plane. These equations and formulæ are not usually given in our American elementary text-books; and yet they are so important in their applications, they enable us to abridge and to simplify the solution of examples to so great an extent, that they should always be taught, even though considerable else may have to be omitted.

To make the student familiar with the principles of the subject, a large number of examples is given at the ends of the chapters, with hints for the solution of the more difficult ones.

In preparing this book, I have consulted freely what works were available to me. Credit could not be given in every case, even if in an elementary text-book it were desirable to do so. In the geometry of two dimensions I am indebted chiefly to the excellent works of Salmon, O'Brien, Todhunter, Puckle, and Howison. In the geometry of three dimensions, my chief indebtedness is to Gregory's Solid Geometry and Salmon's Analytic Geometry of Three Dimensions, while the works of Howison, Church, and Davies have been consulted with advantage. For the Chordel I am indebted to Mr. J. Bruen Miller, of the Class of '79 of this College.

I have to thank my friend and former pupil, Mr. R. W. Prentiss, B. S., of the class of '78, now a Fellow in Mathematics at the Johns Hopkins University, for his kind aid in reading the MS., and for valuable suggestions.

E. A. B.

RUTGERS COLLEGE,
NEW BRUNSWICK, N. J., Jan. 1880.

TABLE OF CONTENTS.

PART I.

ANALYTIC GEOMETRY OF TWO DIMENSIONS.

CHAPTER I.

THE POINT.

ART. PAGE
1. Definitions 1
2. Method of Co-ordinates 1
3. Division of the Subject 1
4. Systems of Co-ordinates in a Plane 1
5. Rectilinear System 2
6. Notation. Definitions. Interpretations 2
7. Distribution of Signs 3
8. Determination of a Point whose Co-ordinates are given 4
9. Distance between Two Given Points 5
10. Signs of Co-ordinates 6
11. Point dividing that Distance in a Given Ratio 8
12. Polar Co-ordinates 9
13. Determination of a Point in Polar Co-ordinates 11
14. Distance between Two Given Points in Polar Co-ordinates 11
15. Definitions. Locus. Constants. Variables 12
16. Constructing Equations 13
17. Independent Variable. Dependent Variable. Functions 14
18. Continuous and Discontinuous Curves 15
19. Branches of Curves 15
20. Symmetrical Curves 15
21. Discussion of Equations 16

CHAPTER II.

THE RIGHT LINE.

ART. PAGE

22. I. Common Form of Equation 28

II. Symmetrical Form of Equation 30

III. Normal Form of Equation 31

IV. Form of Equation referred to Oblique Axes 32

23. Discussion of Equation of First Degree 34

24. Perpendicular from a Given Point upon a Given Line 37

25. Equation of Line through a Given Point 38

26. Equation of Line through Two Given Points 40

27. Angle between Two Given Right Lines 43

28. Equation of Line making Given Angle with Given Line 46

29. Intersection of Two Lines 49

30. Line through Intersection of Two Given Lines 50

31. Polar Equation of Right Line 52

Examples 54

CHAPTER III.

TRANSFORMATION OF CO-ORDINATES.

32. Transformation of Co-ordinates 57

33. To New Axes Parallel to Old 58

34. From One System of Rectangular Axes to Another 58

35. General Formulæ of Transformation 59

36. From Rectangular to Polar Co-ordinates 61

37. From Polar to Rectangular Co-ordinates 61

38. Locus Unchanged by Transformation 62

39. Geometrical Applications 63

CHAPTER IV.

THE CIRCLE.

40. Definition 69

41. Equation of Circle of Given Radius and Centre 69

42. Tangent to the Circle 73

43. Normal to the Circle. Subnormal. Subtangent 75

44. Tangent to the Circle $(x-a)^2+(y-b)^2=r^2$ 76

45. Intersection of Line and Circle 78

ART. PAGE
46. Length of Tangent from a Given Point.................... 79
47. Chord of Contact.................... 81
48. Locus of Intersection of Tangents at Extremities of a Chord. 82
49. Pole and Polar.................... 83
50. Polar Equation of the Circle.................... 84
Examples.................... 85

CHAPTER V.

THE PARABOLA.

51. Definition of a Conic Section.................... 90
52. Definition of a Parabola.................... 91
53. Equation of the Parabola.................... 92
54. Tangent to the Parabola.................... 94
55. Normal to the Parabola.................... 96
56. Property of Tangent to Parabola.................... 96
57. Intersection of Tangent and Focal Perpendicular.................... 97
58. Point of Contact of Tangent.................... 99
59. Equation of Chord of Contact.................... 100
60. Intersection of Tangents at Extremities of a Chord.................... 100
61. Applications to the Parabola.................... 101
62. Diameter of a Curve.................... 102
63. Equation of Parabola.................... 103
64. Parameter of any Diameter.................... 105
65. Equation of Tangent referred to New Axes.................... 105
66. Polar Equation of Parabola.................... 106
67. Focal Chords.................... 107
Examples.................... 109

CHAPTER VI.

THE ELLIPSE.

68. Definition and Construction of Ellipse.................... 112
69. Equation of the Ellipse.................... 113
70. Transformation to New Axes.................... 114
71. Equation referred to Centre as Origin.................... 114
72. Focal Distance of Point on Ellipse.................... 119
73. Circle described on Major Axis.................... 120
74. Tangent to the Ellipse.................... 121
75. Normal to the Ellipse.................... 124

ART. PAGE
76. Intersection of Tangent and Focal Perpendicular........... 125
77. Point of Contact of Tangent........................... 126
78. Chord of Contact................................ 127
79. Intersection of Tangents at Extremities of a Chord 127
80. Diameter of an Ellipse................................ 129
81. Conjugate Diameters................................ 130
82. Tangent Parallel to a Conjugate Diameter................ 131
83. Extremities of a Diameter............................ 132
84. Length of a Semi-Diameter 132
85. Perpendicular on the Tangent.......................... 133
86. Angle between Conjugate Diameters...................... 133
87. Eccentric Angle and Conjugate Diameters.............. 134
88. Supplemental Chords................................ 135
89. Diameters and Supplemental Chords...................... 136
90. Ellipse referred to Conjugate Diameters.................. 137
91. Tangent referred to Conjugate Diameters................. 138
92. Tangents at Extremities of a Chord..................... 139
93. Tangents at Extremities of a Focal Chord................ 139
94. Intersection of Tangents at Right Angles................. 141
95. Rectangle of Focal Perpendiculars...................... 141
96. Polar Equation of Ellipse............................ 142
97. Polar Equation of Ellipse referred to Centre.............. 143
98. Focal Chords.................................. 144
Examples.................................... 145

CHAPTER VII.

THE HYPERBOLA.

99. Definition. Construction 151
100. Equation of the Hyperbola............................ 152
101. Equation with Left-hand Vertex as Origin................ 153
102. Equation referred to Centre 154
103. Focal Distance of a Point............................ 158
104. The Conjugate Hyperbola............................ 160
105. Equation of Conjugate Hyperbola...................... 160
106. Construction of Conjugate Hyperbolas.................. 162
107. Tangent to the Hyperbola............................ 162
108. Normal to the Hyperbola 163
109. Intersection of Tangent and Focal Perpendicular........... 164
110. Point of Contact of Tangent........................... 165
111. Chord of Contact in Hyperbola 166

ART. PAGE
112. Intersection of Tangents at Extremities of a Chord...... . 166
113. Definition of an Asymptote.............................. 167
114. Asymptotes to an Hyperbola.............................. 167
115. Equation of any Diameter................................ 168
116. Conjugate Diameters in the Hyperbola.................... 171
117. Tangent Parallel to a Conjugate Diameter................ 172
118. Extremities of Conjugate Diameters...................... 172
119. Lengths of Conjugate Diameters.......................... 173
120. Length of Central Perpendicular on Tangent............ ... 173
121. Angle between Conjugate Diameters....................... 174
122. Supplemental Chords..................................... 175
123. Equation of Hyperbola referred to Conjugate Diameters.... 176
124. Tangent referred to Conjugate Diameters................. 178
125. Tangents at Extremities of a Chord...................... 178
126. Focal Chords.. 179
127. Intersection of Tangents at Right Angles................ 180
128. Rectangle of Focal Perpendiculars....................... 180
129. Polar Equation of Hyperbola............................. 181
130. Polar Equation referred to the Centre................... 182
131. Properties peculiar to Hyperbola........................ 184
132. Diagonals of Parallelograms on Conjugate Diameters....... 184
133. Equation of Hyperbola referred to Asymptotes............ 185
134. Tangent referred to the Asymptotes...................... 187
135. Intercepts of Secant between Hyperbola and Asymptotes ... 189
136. Parallelogram on the Asymptotes......................... 189
137. Conjugate Diameters referred to Asymptotes.............. 190
138. Extremities of Conjugate Diameters...................... 190
139. Chords Parallel to a Diameter........................... 191
Examples... 192

CHAPTER VIII.

GENERAL EQUATION OF THE SECOND DEGREE.

140. Locus of Equation of the Second Degree.................. 196
141. General Form of the Equation............................ 196
142. First Transformation.................................... 197
143. Second Transformation................................... 199
144. Relations between Coefficients of Transformation........ 203
145. Properties of Transformation............................ 204
146. Non-central Locus....................................... 205
Examples... 207

CHAPTER IX.

HIGHER PLANE CURVES.

ART. PAGE
147. Equations of Higher Plane Curves........ 219
148. Cissoid of Diocles........ 219
149. Equation of the Cissoid........ 220
150. Conchoid of Nicomedes........ 221
151. Equation of the Conchoid........ 222
152. The Witch of Agnesi........ 224
153. Equation of the Witch........ 224
154. Lemniscate of Bernouilli........ 225
155. The Cycloid........ 227
156. Equation of the Cycloid........ 228
157. Equation of the Cycloid........ 229
158. Spirals........ 230
159. The Spiral of Archimedes........ 230
160. Equation of the Spiral of Archimedes........ 231
161. Reciprocal or Hyperbolic Spiral........ 232
162. The Lituus........ 233
162*a*. The Chordel........ 233
163. The Logarithmic Spiral........ 235

PART II.

ANALYTIC GEOMETRY OF THREE DIMENSIONS.

CHAPTER I.

THE POINT.

164. Determination of a Point in Space........ 237
165. Co-ordinate Planes and Axes........ 238
166. Positions of Co-ordinate Planes........ 239
167. Convention as to Signs........ 239
168. Theory of Projections........ 240
169. Distance between Two Points........ 241
170. Direction Cosines of a Radius-Vector........ 242
171. Point in Polar Co-ordinates........ 243

CHAPTER II.

THE RIGHT LINE.

ART. PAGE
172. Equations of a Right Line........ 244
173. Line in Space under Certain Conditions........ 246
174. Angle between Two Lines in Space........ 247
175. Angle in Terms of Special Constants........ 248
176. Intersection of Two Lines........ 250
Examples........ 252

CHAPTER III.

THE PLANE.

177. Equation of a Plane........ 255
178. Traces of a Plane........ 256
179. The Equation of the First Degree........ 257
180. Equations in Terms of Intercepts........ 258
181. Perpendicular on a Plane from the Origin........ 258
182. Length of Perpendicular from a Given Point........ 260
183. Angle between Two Planes........ 261
184. Angle between a Line and a Plane........ 263
Examples........ 264

CHAPTER IV.

SURFACES OF REVOLUTION.

185. Definition. Meridian Sections and Planes........ 268
186. General Equation of a Surface of Revolution........ 268
187. Equation of a Right Circular Cone........ 269
188. Equation of a Cylinder........ 270
189. Equation of a Sphere........ 270
190. Equation of a Paraboloid of Revolution........ 271
191. Ellipsoids of Revolution........ 272
192. The Prolate Spheroid. Note........ 272
193. The Oblate Spheroid........ 275
194 Hyperboloids of Revolution........ 275
195. Hyperboloid of Revolution of Two Nappes........ 275
196. Hyperboloid of Revolution of One Nappe........ 276
197. Sections of a Cone........ 278
198. Tangent Plane to an Ellipsoid........ 281
Examples........ 284

PART I.

ANALYTIC GEOMETRY OF TWO DIMENSIONS.

CHAPTER I.

THE POINT.

1. Analytic Geometry is that branch of Mathematics in which the magnitudes considered are represented by letters, and the properties and relations of these magnitudes are investigated by the aid of Algebraic Analysis.

2. All the points of the magnitudes to be considered are referred to fixed objects, by means of elements called co-ordinates, and hence this method is sometimes known as **Co-ordinate Geometry**. It was introduced by Descartes in 1637, and hence is also called the **Cartesian System.**

3. Analytic Geometry is divided into two parts: **Analytic Geometry of two Dimensions**, which treats of lines lying wholly in a single plane, and requires but two co-ordinates to determine the position of a point; and **Analytic Geometry of three Dimensions**, which treats of lines and surfaces lying in any manner in space, and requires three co-ordinates to determine the position of a point.

4. There are two systems of co-ordinates in common use for determining the position of a point in a plane. The first is by means of its distances from any two given right lines of the plane which intersect each other. The second is by means of its distance and direction from a given point in the plane. The first is called the **Rectilinear System**, and the second is called the **Polar System.**

5. Let us suppose that we have given the position of two fixed right lines XX′, YY′, intersecting in the point O, and let the plane of the two lines be represented by the surface of the paper. Now, if through any point P we draw PM parallel to OY, and PN parallel to OX, it is plain that the position of P is known if the lengths of PM and PN are known. For example, if we have given PN $= a$, PM $= b$, we can determine the position of the point P with regard to the lines OX and OY: we need only measure OM $(= a)$ along OX, and ON $(= b)$ along OY, and draw the parallels PM, PN: P will be the point whose position we wished to determine.

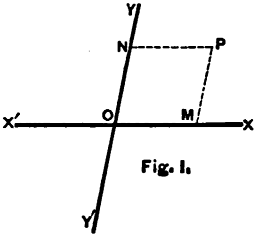

Fig. 1.

6. The line PM, or its equal ON, is usually denoted by the letter y, and is called the **Ordinate** of the point P. OM, or its equal NP, is denoted by the letter x, and is called the **Abscissa** of the same point; and the two lines, when spoken of together, are called the **Co-ordinates** of P.

The lines XX′ and YY′ are called the **Axes of Co-ordinates**, or the **Co-ordinate Axes**, and the point O in which they intersect is called the **Origin**. The line XX′ is called the **Axis of Abscissas**, or the **Axis of x.** It may have any direction, but it is usually assumed to be horizontal. The line YY′ is called the **Axis of Ordinates**, or the **Axis of y.** The axes are said to be rectangular or oblique, according as the angle at which they intersect is a right or an oblique angle. The rectangular axes are the most simple, and, in this work, will always be employed, unless otherwise specified.

The abscissa of any point is its distance from the axis of ordinates, measured on a line parallel to the axis of abscissas.

The ordinate of any point is its distance from the axis of abscissas, measured on a line parallel to the axis of ordinates.

The point P is said to be determined when the values of its co-ordinates, x and y, are given, as by the two equations $x = a$, $y = b$. For example, if we have given that $x = 5$ feet, $y = 3$ feet, we shall determine the position of the point of which x and y are the co-ordinates, by measuring, from the origin O, on the axis of x, a distance OM equal to 5 feet; then through M draw a line parallel to the axis of y, and on this line measure a distance MP equal to 3 feet. P will be the position of the point required.

Hence, in order to determine the position of a point, it is sufficient to have the two equations, $x = a$, $y = b$, in which a and b are given. These equations are the analytic representatives of the point, and are called the **Equations of a Point.**

It will easily be seen that the equations of the point M, in the preceding figure, are $x = a$, $y = 0$; that those of the point N are $x = 0$, $y = b$; and of the origin itself are $x = 0$, $y = 0$. The point whose position is defined by the equations $x = a$, $y = b$, is commonly spoken of as the point (a, b).

7. In order that the equations $x = a$, $y = b$, should be satisfied by only one point, it is necessary to pay attention, not only to the absolute values of the co-ordinates, but also to the signs of these quantities.

If no attention were given to the signs of the co-ordinates, we might measure $OM = a$, and $ON = b$ (Fig. 2), on either side of the origin, and any of the four points, P, P′, P″, P‴, would satisfy the equations $x=a$, $y=b$. This ambiguity, however, may be avoided by distinguishing algebraically between the lines OM and OM′, by giving them different signs. If lines measured in one direction be considered as positive, lines measured in the opposite direction must be considered negative. It is, of course, arbitrary in which direction we measure positive lines; but it is customary to regard OM measured towards the right, and ON measured upwards, as positive; and hence, OM′ and ON′, measured in the opposite directions, must be considered negative, as in Trigonometry.

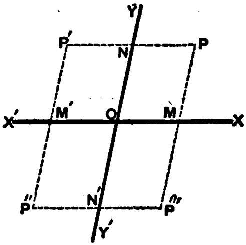

Fig. 2.

The four angles into which the plane is divided by the axes are distinguished thus: The angle YOX is called the **First Angle**; YOX′, the **Second Angle**; Y′OX′, the **Third Angle**; and Y′OX the **Fourth Angle**. If P, P′, P″, P‴ be points situated in the four angles, they will be represented by the following equations:

$$\text{P}\left\{\begin{array}{l} x=a, \\ y=b. \end{array}\right. \qquad \text{P}''\left\{\begin{array}{l} x=-a, \\ y=-b. \end{array}\right.$$

$$\text{P}'\left\{\begin{array}{l} x=-a, \\ y=b. \end{array}\right. \qquad \text{P}'''\left\{\begin{array}{l} x=a, \\ y=-b. \end{array}\right.$$

Or, by (a, b), $(-a, b)$, $(-a, -b)$, $(a, -b)$ respectively.

8. *To determine a point whose co-ordinates are given.*

Lay off from the origin, on the axis of x, a distance equal to the given abscissa, to the right if the abscissa is +, and to the left if it is —. Through the point thus found, draw a line parallel to the axis of y, and lay off on it a distance from the axis of x equal to the given ordinate, above if the ordinate is +, and below if it is —. The point thus found will be the position of the point required.

EXAMPLES.

1. Find the point whose co-ordinates are (4, 3).

Draw the axes XX′ and YY′. Lay off OA to the right of the origin, equal to 4 units of a given scale; draw AB parallel to YY′, and lay off AP upward and equal to 3 units of the same scale. P is the point required.

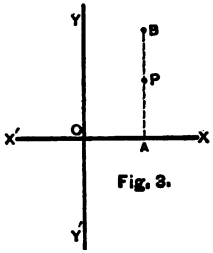

Fig. 3.

2. Find the points whose co-ordinates are (— 2, — 3), (4, — 3), (— 5, 0), (— 8, 5), (11, 3).

3. Construct the triangle whose vertices are (0, 0), (2, — 3), (— 1, 0).

4. Construct the figure whose vertices are (2, 9), (— 8, 9), (— 8, 1), (2, 1).

9. *To find a formula for the distance between two points whose co-ordinates are known.*

1st. *With regard to rectangular axes.* Let P and Q be the two points, x', y' the co-ordinates of P, x'', y'' the co-ordinates of Q, and the distance PQ be represented by d. Then, from Geometry, we have,

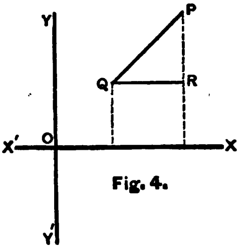

Fig. 4.

$$\overline{PQ}^2 = \overline{QR}^2 + \overline{PR}^2,$$

or $$PQ = \sqrt{\overline{PR}^2 + \overline{QR}^2};$$

but $PR = y' - y''$, and $QR = x' - x''$.

Hence, $$d = \sqrt{(x' - x'')^2 + (y' - y'')^2}. \qquad (1)$$

2d. *With regard to oblique axes.* Let ω represent the angle YOX, and the points P and Q be represented as before. From Trigonometry we have,

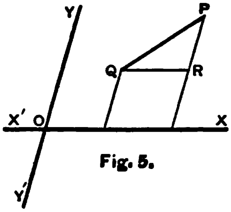

Fig. 5.

$$\overline{PQ}^2 = \overline{PR}^2 + \overline{QR}^2 - 2QR \cdot PR \cos QRP,$$

Or, $$PQ = \sqrt{\overline{PR}^2 + \overline{QR}^2 - 2PR \cdot QR \cos QRP}.$$

But $PQ = d$; $QR = x' - x''$;

$PR = y' - y''$; $QRP = \pi - \omega$.

Hence,

$$d = \sqrt{(x'-x'')^2 + (y'-y'')^2 + 2(x'-x'')(y'-y'')\cos\omega}. \quad (2)$$

COR.—If the axes are rectangular, $\omega = 90°$, and hence $\cos\omega = 0$. Hence,

$$d = \sqrt{(x'-x'')^2 + (y'-y'')^2},$$

which is the same as (1), as it ought to be.

If one of the points, as Q, were the origin, so that $x'' = 0$, $y'' = 0$, we would have, from (1),

$$d = \sqrt{x'^2 + y'^2},$$

and from (2), $$d = \sqrt{x'^2 + y'^2 + 2x'y' \cos\omega}.$$

10. In using these formulæ, attention must be given to the *signs* of the co-ordinates. If the point Q, for instance, be in the angle XOY′, the sign of its ordinate y'' will be negative, and the line PR will be the *sum* and not the *difference* of y' and y''; and we must write $y' + y''$ instead of $y' - y''$ in the formulæ.

EXAMPLES.

1. Find the distance from the point $(-8, -2)$ to the point $(3, 7)$.

Ans. $d = \sqrt{(3+8)^2+(7+2)^2} = \sqrt{121+81} = 14.21.$

2. Find the distance between the two points $(2, -3)$ and $(-5, 6)$, the axes being inclined at an angle of 60°.

Here $x' - x'' = 2 + 5 = 7$; $y' - y'' = -3 - 6 = -9$; and $\cos\omega = \frac{1}{2}$. Hence, in (2) we get

$$d = \sqrt{49+81-2\cdot 7\cdot 9\ \tfrac{1}{2}} = \sqrt{49+81-63} = \sqrt{67}.$$

3. Find the lengths of the sides of a triangle, the co-ordinates of whose vertices are $(2, 3)$, $(4, -5)$, $(-3, -6)$.

Ans. $\sqrt{68}$, $\sqrt{50}$, $\sqrt{106}$.

4. Find the lengths of the sides of a triangle, the co-ordinates of whose vertices are the same as in the last example, the axes being inclined at an angle of 60°.

Ans. $\sqrt{52}$, $\sqrt{57}$, $\sqrt{151}$.

5. Find the lengths of the three sides of the triangle whose vertices are $(2, 5)$, $(-4, 1)$, $(-2, -6)$.

Ans. $\sqrt{52}$, $\sqrt{53}$, $\sqrt{137}$.

6. Express algebraically that the distance of the point (x, y) from the point $(2, 3)$ is equal to 4.

Ans. $\sqrt{(x-2)^2 + (y-3)^2} = 4.$

7. Express algebraically that the point (x, y) is equidistant from the points $(2, 3)$ and $(4, 5)$.

Ans. $(x-2)^2 + (y-3)^2 = (x-4)^2 + (y-5)^2,$
or $x + y = 7.$

8. Find the point equidistant from the points $(2, 3)$, $(4, 5)$, $(6, 1)$.

Here we have two equations, formed as in Ex. 7, to determine the two unknown quantities.

Ans. $x = \frac{13}{3}$, $y = \frac{8}{3}$, and the common distance is $\frac{\sqrt{50}}{3}$.

11. *To find the co-ordinates of the point which divides in a given ratio, $m : n$, the right line joining two given points, (x', y') and (x'', y'').*

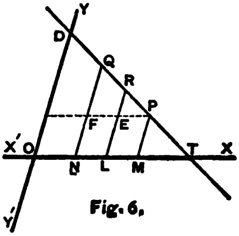

Fig. 6.

Let P and Q be the two given points, (x', y') and (x'', y''), and R the required point, whose co-ordinates we denote by x and y. Then we have,

$$\text{PR} : \text{RQ} :: m : n.$$

Draw the ordinates PM, RL, QN, and the line PEF parallel to OX; then we have,

$$\frac{\text{PR}}{\text{RQ}} = \frac{\text{PE}}{\text{EF}} = \frac{\text{ML}}{\text{LN}} = \frac{m}{n},$$

or,
$$\frac{m}{n} = \frac{x' - x}{x - x''};$$

hence,
$$x = \frac{mx'' + nx'}{m + n}.$$

Similarly we have,
$$y = \frac{my'' + ny'}{m + n}.$$

If the line were to be cut *externally* in the given ratio, we should have,

$$m : n :: x - x' : x - x'',$$

and therefore,

$$x = \frac{mx'' - nx'}{m - n}, \qquad y = \frac{my'' - ny'}{m - n}.$$

If $m = n$, or PQ is bisected in R, we have,

$$x = \frac{x'' + x'}{2}, \qquad y = \frac{y'' + y'}{2},$$

a result which is of frequent use. In this article the axes may be oblique or rectangular, the result being the same.

EXAMPLES.

1. Find the co-ordinates of the middle points of the sides of the triangle whose vertices are (2, 3), (4, —5), (—3, —6). *Ans.* $(\frac{1}{2}, -\frac{11}{2})$, $(-\frac{1}{2}, -\frac{3}{2})$, $(3, -1)$.

2. The line joining the points (2, 3), (4, — 5), is trisected; to find the co-ordinates of the point of trisection nearest to the former point. *Ans.* $x = \frac{8}{3}$, $y = \frac{1}{3}$.

3. The co-ordinates of P are (2, 3), and of Q (3, 4); find the co-ordinates of R, so that PR : RQ :: 3 : 4. *Ans.* $x = 2\frac{3}{7}$, $y = 3\frac{3}{7}$.

4. The point (x, y) is midway between (3, 4) and (—5, —8); find its distance from the origin. *Ans.* $\sqrt{5}$.

POLAR CO-ORDINATES.

12. Let O be a given point, and OA a fixed line through it; it is evident that we shall know the position of any point P, if we know the length OP and the angle POA. The line OA is called the **Initial Line** (called also the **Prime Radius** and the **Polar Axis**), the fixed point is called the **Pole**, the line OP is called the **Radius Vector**, and the variable angle AOP is called the **Direction Angle**, or **Vectorial Angle**. This method is called the method of **Polar Co-ordinates**. The *initial line* may have any position in the plane, but it is usually drawn through O horizontally to the right. The angle AOP and the distance OP are the polar co-ordinates of P.

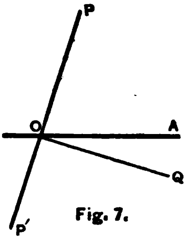

Fig. 7.

If the *direction angle* of any point be denoted by θ, and its *radius vector* by ρ, the point may be called the point (ρ, θ). When the *direction angle* is estimated from A *upwards* towards P, as in Trigonometry, it is called +;

when estimated in the *opposite* direction from A *downwards* towards Q, it is called —. The *radius vector* is + when estimated from the pole in the direction of the extremity of the arc which measures the direction angle; and it is — when estimated in the *opposite* direction.

The following example* will make this clear. Let a be any distance OP, measured from O towards P, θ being the angle which OP makes with OA; then

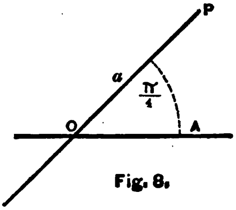

Fig. 8.

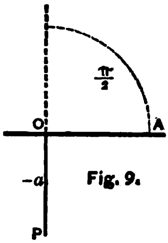

Fig. 9.

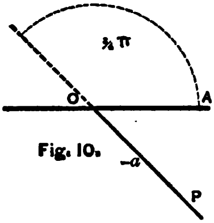

Fig. 10.

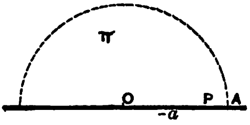

Fig. 11.

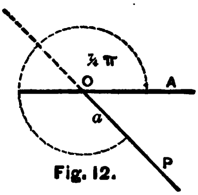

Fig. 12.

$\theta = \frac{1}{4}\pi, \quad \rho = a,$ represents P in Fig. 8;
$\theta = \frac{1}{2}\pi, \quad \rho = -a,$ " " " 9;
$\theta = \frac{3}{4}\pi, \quad \rho = -a,$ " " " 10;
$\theta = \pi, \quad \rho = -a$ " " " 11;
$\theta = \frac{7}{4}\pi, \quad \rho = a,$ " " " 12;

We observe that the direction in which ρ is measured depends, not only on its sign, but also on the value of θ; thus, when $\theta = \frac{3}{4}\pi$, and $\rho = -a$, ρ must be measured from O to P, as in Fig. 3; and when $\theta = \frac{7}{4}\pi$, $\rho = a$, ρ must be measured in exactly the same direction.

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* Puckle's Conic Sections, p. 9. Also, O'Brien's Co-ordinate Geometry, p. 37.

13. *To locate a point whose polar co-ordinates are given.*

Draw the initial line, and lay off, at any point taken for the pole, an angle equal to the given angle θ; then measure the distance ρ from the pole, in the direction of the extremity of the arc which measures the direction angle, or in the opposite direction, according as ρ is + or −, and the required point is obtained.

EXAMPLES.

1. Locate the point $\rho = 7$, $\theta = \frac{1}{4}\pi$.

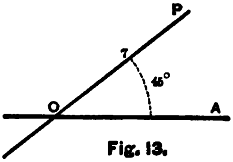


Fig. 13.

The radius of the measuring arc being 1, π is the semi-circumference. Hence, $\frac{1}{4}\pi = 45°$. Now draw the initial line OA, and, at the point O taken for the pole, lay off AOP = 45°, and measure OP = + 7; P is the point required.

2. Represent the points $\rho = -8$, $\theta = \pi$, and $\rho = 15$, $\theta = \frac{1}{2}\pi$.

3. Represent the points $\rho = 15$, $\theta = \frac{3}{2}\pi$, and $\rho = -6$, $\theta = \frac{5}{4}\pi$.

4. Represent the points $\rho = -6$, $\theta = -\frac{5}{6}\pi$, and $\rho = 10$, $\theta = \frac{1}{4}\pi$.

5. Represent the points $\rho = 5$, $\theta = \frac{2}{3}\pi$, and $\rho = 6$, $\theta = \frac{1}{6}\pi$.

14. *To find the distance between two points in terms of their polar co-ordinates.*

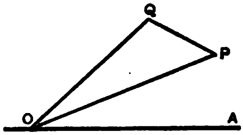


Fig. 14.

Let P and Q be the two points; represent the co-ordinates of P by ρ', θ', and of Q by ρ'', θ'', and the distance PQ by d. Then in the triangle OPQ, OP $= \rho'$, OQ $= \rho''$, and the angle $POQ = \theta'' - \theta'$. Hence, from Trigonometry we have,

$$d = \sqrt{\rho''^2 + \rho'^2 - 2\rho''\rho' \cos(\theta'' - \theta')}. \qquad (1)$$

Cor.—If $\rho'' = 0$, and $\theta'' = 0$, we have, for the distance of any point (ρ', θ') from the origin, $d = \rho'$.

EXAMPLES.

1. Find the distance between $\rho = 3$, $\theta = \frac{1}{3}\pi$, and $\rho = 4$, $\theta = \frac{5}{6}\pi$. *Ans.* 5.

2. Find the distance between $\rho = 5$, $\theta = 75°$, and $\rho = 4$, $\theta = 15°$. *Ans.* $\sqrt{21}$.

3. Find the distance between $\rho = 8$, $\theta = \frac{3}{4}\pi$, and $\rho = 3$, $\theta = \frac{13}{12}\pi$. *Ans.* 7.

4. Find the distance between (5, 30°) and (6, 225°). *Ans.* 109.

DEFINITIONS.

15. The **Equation of a Line** is the equation which expresses the relation between the co-ordinates of every point of the line.

The term **Locus** is nearly synonymous with **Geometric Figure**; it is the series of positions to which a point or line is restricted by given conditions.

The **Locus of a Point** is the line generated by the point when moving according to some given law.

The **Locus of a Line** is the surface generated by that line when moving according to a given law.

The **Locus of an Equation** is the line or surface, the co-ordinates of all of whose points satisfy the equation, while the equation is the *analytic representative* of the line or surface. Thus, take the equation,

$$y = x + 4.$$

We may assign to x any value we please, and, from the equation, determine the corresponding value of y. Thus, corresponding to the values 1, 2, 3, 4, etc., for x, we have 5, 6, 7, 8, etc., for y. Now, if we suppose a line that passes through every point determined by giving to x and y values that satisfy the equation,

$$y = x + 4,$$

this line is called the *locus of the equation.* Since the *locus* is the geometric equivalent of the equation, every equation between variables which denote the co-ordinates of a point, represents a locus.

There are two kinds of quantities used in Analytic Geometry: 1st, **Constants**, whose values do not change in the same discussion, and are represented by the leading letters of the alphabet; and 2d, **Variables**, which are susceptible of an infinite number of values within certain limits that are determined by the nature of the problem, and are represented by the final letters of the alphabet.

CONSTRUCTING EQUATIONS.

16. To construct an equation, or find its locus, is to trace, by means of determined points, the geometric figure which the equation represents.

To construct any curve from its equation, we solve the equation for either of its variables, usually for y, whose value or values we find in terms of x and constants. Then substitute for x a series of arbitrary values, and find the corresponding values of y. Now draw the axes, and lay down the points corresponding to the co-ordinates thus found. A curve traced through these points will approximately represent the locus of the equation. The closer the points are to each other, the more exact is the locus, unless it be a right line, which needs but two points to determine it.

SCH.—Although it is customary to solve the equation for y, yet if it is above the second degree with respect to either variable, it is expedient to solve it with respect to the variable which is least involved. Thus, to construct

$$2x + y^2 = 3y^3 + 2y - 8,$$

we solve it with respect to x, and find

$$x = \frac{3y^3 + 2y - 8 - y^2}{2}.$$

Then substitute arbitrary values for y, and find the corresponding values of x.

17. The **Independent Variable** is the one to which arbitrary values are assigned, usually x. The other is called the **Dependent Variable.** This distinction is made for convenience; either variable may be treated as the *independent variable,* and the other as the *dependent variable;* the latter is said to be a **Function** of the former.

One quantity is a function of another when so connected with it that no change can take place in the latter, without producing a corresponding change in the former. Thus,

$$y = ax + b,$$

y is a function of x; the ordinate of a curve is a function of the abscissa.

Functions are divided into two classes, *algebraic* and *transcendental.*

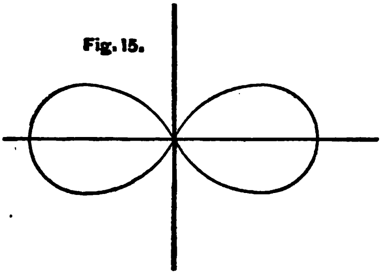
An **Algebraic Function** is one in which the relation between the function and its variable can be expressed by the ordinary operations of algebra, that is, *addition, subtraction, multiplication, division, involution,* and *evolution,* or the algebraic sum of many such functions. Thus, in each of the following expressions,

$$y = 2x^2 + x^3, \quad y = 4x - \sqrt{x}, \quad y = (ax^3 - 2x^2)^{\frac{1}{3}},$$

y is an algebraic function of x.

A **Transcendental* Function** is one in which the relation between the function and its variable cannot be expressed by the ordinary operations of algebra.

Fig. 15.



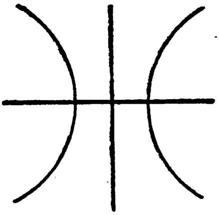


Fig. 16.

18. A curve is **Continuous** when it has no interruption in its extent, and no abrupt change in its curvature. A circle, an ellipse, and the curve in Fig. 15 are examples of *continuous* curves. The curve, Fig. 16, is *discontinuous*, having an interruption in *extent*. Fig. 17 is an example of a *discontinuous* curve, having an abrupt change in its curvature.

19. A **Branch** is the continuous part of a curve. In Figs. 16 and 17 the curves have two branches.

20. A curve is **Symmetrical** with respect to any line when it has the same form on both sides of the line; that is, when every point on one side of the line has a corresponding point on the other side of the line. The curves in Figs. 15 and 16 are symmetrical with respect to the axis of x. The curve in Fig. 17 is not symmetrical with respect to any line.

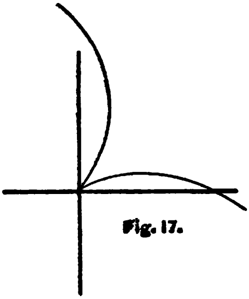


Fig. 17.

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* Transcendental functions are further subdivided, but this division is not necessary in the present work.

DISCUSSION OF EQUATIONS.

21. The **Discussion** of an equation consists in observing the peculiarities of the loci which appear from the form of the equation, by making different hypotheses on the quantities that enter it.

1st. *To find where the locus cuts the axis of x.*

Make $y = 0$ in the equation, and find the corresponding value of x, which will be the intersection with the axis of x. Similarly, make $x = 0$, and find the corresponding value of y, which will be the intersection with the axis of y.

Thus, in the locus,

$$y = x + 2,$$

if $x = 0$, $y = 2$, and if $y = 0$, $x = -2$. Therefore the locus cuts the axis of x at the distance 2 to the left of the origin, and the axis of y at the distance 2 above the origin.

2d. *To find the limits between which the locus is situated, and to test for continuity in extent.*

The limits are discovered by determining the greatest and least values of the independent variable which give *real* values to the dependent one. If all values assigned to x between certain limits give rise to *real* values for y, the corresponding points will be *real;* that is, the curve will be *continuous* in extent between these limits. If, on the contrary, there are certain values of x which render y *imaginary*, the corresponding points will be *imaginary;* that is, the locus is *interrupted* at such points, and therefore is *discontinuous.* If between any *two* values of either variable, the corresponding values of the other variable are *all* imaginary, the locus does not exist between the corresponding limits. And the limits of discontinuity are the limits between which the values of the dependent variable are imaginary.

Thus, in the locus,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

by solving for y we obtain,

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2};$$

so that y is *real* for every value of x that lies *beyond* the limits $x = a$ and $x = -a$, but is *imaginary* for every value of x lying *between* them; and the locus is interrupted in the latter region.

3d. *To test for symmetry with respect to an axis.*

Note whether, for each *real* value of one variable, the other has *two* values numerically equal, but with contrary signs; if so, there are points similarly situated on opposite sides of the axis from which the variable, having two values, is reckoned; and hence the locus is symmetrical with respect to that axis.

Thus, in the locus last considered, we see that for any value of x beyond the limits $+a$ and $-a$, y has two values numerically equal, with contrary signs. Hence this locus is symmetrical with respect to the axis of x.

EXAMPLES.

1. Construct and discuss the equation

$$2y - 6x - 12 = 0.$$

Solving the equation for y, we have,

$$y = 3x + 6.$$

Making successively $x = 0$ and $y = 0$, we obtain,

$$y = 6, \qquad \text{and} \qquad x = -2.$$

The locus, therefore, cuts the axis of x at a distance 2 to the left of the origin, and the axis of y at a distance 6 above the origin. Draw the axes XX′ and YY′, and lay down the corresponding points.

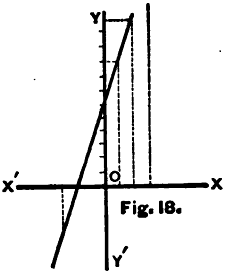


Fig. 18.

Now, give x the following arbitrary values, and find the corresponding values of y:

When $x = 1$, $y = 9$, giving the point $(1, 9)$.
" $x = 2$, $y = 12$, " " $(2, 12)$.
" $x = 3$, $y = 15$, " " $(3, 15)$.

All positive values of x give *positive, real,* and *single* values to y. The equation being of the first degree, the locus has but one branch, which extends to the right of the axis of y indefinitely, and above the axis of x.

Giving negative values to x, we have:

When $x = -1$, $y = 3$, giving the point $(-1, 3)$.
" $x = -2$, $y = 0$, " " $(-2, 0)$.
" $x = -3$, $y = -3$, " " $(-3, -3)$.

For all subsequent negative values of x, y has *real, negative,* and *single* values. Hence, the locus has a single branch extending indefinitely in the third angle. Laying down the points $(-3, -3)$, $(-2, 0)$, $(-1, 3)$, $(1, 9)$, $(2, 12)$, $(3, 15)$, we find that they all come upon the right line drawn through $(0, 6)$ and $(-2, 0)$, which is therefore the locus represented by the given equation.

If any other values be assigned to x, either positive or negative, integral or fractional, and the corresponding values of y be deduced, the points so determined will all fall upon the same line.

2. Construct and discuss the equation $\frac{1}{3}y + x = 2$.

Result.—A straight line cutting the axis of x at $(2, 0)$, and the axis of y at $(0, 6)$.

3. Construct and discuss the equation

$$8y - 6x - 5 = 0.$$

Result.—A right line passing through the points $(0, \frac{5}{8})$ and $(-\frac{5}{6}, 0)$.

4. Construct and discuss the equation

$$x^2 + y^2 = 16.$$

Solving the equation for y, we get

$$y = \pm\sqrt{16 - x^2}.$$

When $x = 0$, $y = +4$ and -4.

Hence the locus cuts the axis of y at $(0, 4)$ and $(0, -4)$.

When $y = 0$, $x = +4$ and -4.

Hence the locus cuts the axis of x at $(4, 0)$ and $(-4, 0)$.

As every value of x between $+4$ and -4 gives two *real* values for y, numerically equal, with contrary signs, the locus is symmetrical with respect to the axis of x, and continuous between these limits. But when $x > 4$ or < -4, y becomes imaginary, and therefore the locus has no point beyond its intersection with the axis of x.

Similarly, $x = \pm\sqrt{16 - y^2}$ shows that the locus is symmetrical with respect to the axis of y, and continuous between $y = 4$ and -4. When y is > 4 or < -4, the values of x become *imaginary;* and hence the locus has no point beyond its intersection with the axis of y.

Now giving to x arbitrary values between $+4$ and -4, we find the following:

When $x = 1$, $y = \pm 3.9$ nearly, giving us the points $(1, 3.9)$ and $(1, -3.9)$.

When $x = 2$, $y = \pm 3.5$ nearly, giving us the points $(2, 3.5)$ and $(2, -3.5)$.

When $x = 3$, $y = \pm 2.6$ nearly, giving us the points $(3, 2.6)$ and $(3, -2.6)$.

The negative values of x give us the following points:
$(-1, 3.9)$ and $(-1, -3.9)$,
$(-2, 3.5)$ and $(-2, -3.5)$,
$(-3, 2.6)$ and $(-3, -2.6)$.

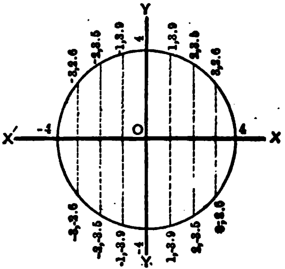


Fig. 19.

Constructing the points thus found, we find the figure to be the circumference of a circle whose radius is 4, and which is symmetrical to both axes. If any fractional values be given to x between the limits $+4$ and -4, and the corresponding values of y be found, the points so determined will all fall upon the same circumference.

The same result might have been reached by considering that $x^2 + y^2 = 16$ shows that the distance of any point (x, y) from the origin is constantly equal to 4. (Art. 9.)

5. Construct and discuss the equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

Solving the equation for y, we get,

$$y = \pm \tfrac{2}{3}\sqrt{9 - x^2}.$$

Making $x = 0$, we get $y = \pm 2$; hence the curve cuts the axis of y in two points, one at the distance 2 above the origin, and the other at the same distance below it.

Making $y = 0$, we get $x = \pm 3$; hence the curve cuts the axis of x in two points equally distant from the origin and on opposite sides of it. For each value of x between $+3$ and -3, y is *real*, and has two values numerically equal, with contrary signs; hence the curve is symmetrical with respect to the axis of x, and continuous between these limits. When $x > 3$ or < -3, y becomes imaginary, and hence the curve has no point beyond its intersection with the axis of x.

Similarly, from $x = \pm \frac{3}{2}\sqrt{4 - y^2}$, we learn that the curve is continuous between $y = 2$ and $y = -2$, and symmetrical with respect to the axis of y, but has no point beyond its intersection with the axis of y. Since the curve is symmetrical with respect to the axis of y, we need consider only positive values of x.

Giving now to x the following values, we have the following corresponding values for y:

When $x = 0$, $y = \pm 2$, giving the points $(0, 2)$ and $(0, -2)$.

When $x = .5$, $y = \pm 1.97$, giving the points $(.5, 1.97)$ and $(.5, -1.97)$.

When $x = 1$, $y = \pm 1.89$, giving the points $(1, 1.89)$ and $(1, -1.89)$.

When $x = 1.5$, $y = \pm 1.73$, giving the points $(1.5, 1.73)$ and $(1.5, -1.73)$.

When $x = 2$, $y = \pm 1.49$, giving the points $(2, 1.49)$ and $(2, -1.49)$.

When $x = 2.5$, $y = \pm 1.1$, giving the points $(2.5, 1.1)$ and $(2.5, -1.1)$.

When $x = 2.75$, $y = \pm 0.8$, giving the points $(2.75, 0.8)$ and $(2.75, -0.8)$.

When $x = 3$, $y = 0$, giving the points $(3, +0)$ and $(3, -0)$.

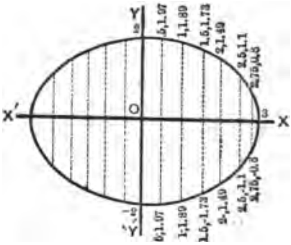


Fig. 20.

Laying down the points thus found, and a similar set on the left of the axis of y, we determine the figure to be an ellipse, whose axes are 6 and 4.

6. Construct and discuss the equation $y^2 = 3x - 9$.

Ans. It cuts the axis of x at the distance 3 to the right of the origin, and lies entirely to the right of this point, extending indefinitely in two branches that are symmetrical with respect to the axis of x. The curve is a parabola.

7. Construct and discuss the equation

$$x^2 - y^2 = 16.$$

Ans. The curve cuts the axis of x at two points; one at the distance 4 to the right of the origin, and the other at the same distance to the left of it. It has no point between these intersections, but extends to the right and left of these points indefinitely, and is symmetrical with respect to both axes, and is known as the hyperbola.

8. Construct and discuss the following equations:

$$y^2 + x^2 - 6x + 10y + 9 = 0;$$

$$16(y + 3)^2 = 200 - 25(x - 5)^2;$$

$$y^2 = x^4 - x^3.$$

9. Construct and discuss the equation

$$x = \log y \qquad \text{or} \qquad y = a^x.$$

Assuming $a = 10$, which is the base of the common system, and giving to x the series of values in the following table, the values of y can be found from a table of logarithms.

When $x = 0$, $y = 1.$
" $x = .2$, $y = 1.58$ nearly.
" $x = .4$, $y = 2.51$ "
" $x = .6$, $y = 3.98$ "
" $x = .8$, $y = 6.31$ "
" $x = 1$, $y = 10.00.$
" $x = -.1$, $y = .8$ nearly.
" $x = -.2$, $y = .6$ "
" $x = -.4$, $y = .4$ "
" $x = -.7$, $y = .2$ "
" $x = -1$, $y = .1$ "
" $x = -2$, $y = .01.$

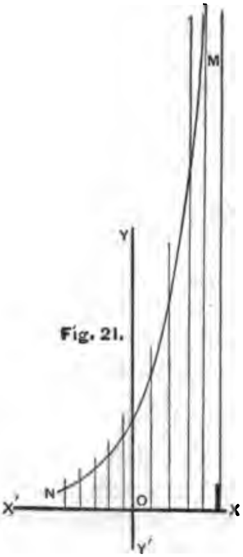


Fig. 21.

Laying down the points thus found, we have the curve MN (Fig. 21), which is called the *logarithmic curve.* It lies entirely above the axis of x, since negative numbers have no logarithms, but extends on both sides of the axis of y indefinitely, and cuts it at (0, 1), through which point all logarithmic curves pass, since log. 1 = 0 in any system. The curve is not symmetrical with respect to the axis of y; as it continues to the left of the origin, the ordinates diminish more and more, but can never reduce to 0 while x is finite. When the ordinate becomes infinitely small, the abscissa becomes infinitely great, and negative.

10. Construct and discuss $x = \log y$ or $y = a^x$

Assume $a = 2.718$, which is the base of the Naperian system, and we get $y = (2.718)^x$.

When $x = 1$, $y = 2.718$.
" $x = 2$, $y = 7.389$.
" $x = 3$, $y = 20.085$.
" $x = -1$, $y = 0.368$.
" $x = -2$, $y = 0.135$.

Laying down the points thus found, we have the curve MN (Fig. 22), which is called the logarithmic curve for the Naperian base.

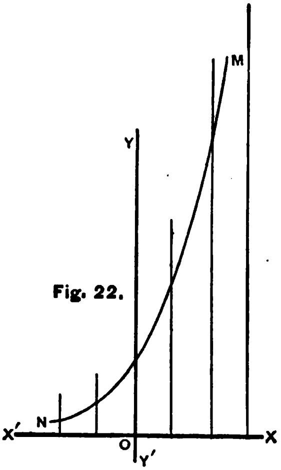


Fig. 22.

11. Construct and discuss the equation of the sinusoid

$$y = \sin x.$$

The unit of angular measure is the angle at the centre, measured by an arc equal in length to the radius, as this angle is of an invariable magnitude, whatever be the length of the radius. The semi-circumference being 3.1416, when the radius is unity, the number of degrees in an arc equal to the length of the radius is equal to $\frac{180°}{3.1416} = 57°.3$ nearly. Hence the following series of values:

When $x = 0° = 0,$ $y = 0.$
" $x = 10° = .17,$ $y = .17.$
" $x = 20° = .35,$ $y = .34.$
" $x = 30° = .52,$ $y = .50.$
" $x = 40° = .70,$ $y = .64.$
" $x = 50° = .87,$ $y = .77.$
" $x = 60° = 1.05,$ $y = .87.$
" $x = 70° = 1.22,$ $y = .94.$
" $x = 80° = 1.40,$ $y = .98.$
" $x = 90° = 1.57,$ $y = 1.00.$
" $x = 180° = 3.14,$ $y = 0.$
" $x = 190° = 3.31,$ $y = -.17.$
" $x = 200° = 3.49,$ $y = -.34.$
" $x = 210° = 3.66,$ $y = -.50.$
" $x = 220° = 3.84,$ $y = -.64.$
" $x = 230° = 4.01,$ $y = -.77.$
" $x = 240° = 4.19,$ $y = -.87.$
" $x = 250° = 4.36,$ $y = -.94.$
" $x = 260° = 4.54,$ $y = -.98.$
" $x = 270° = 4.71,$ $y = -1.00.$

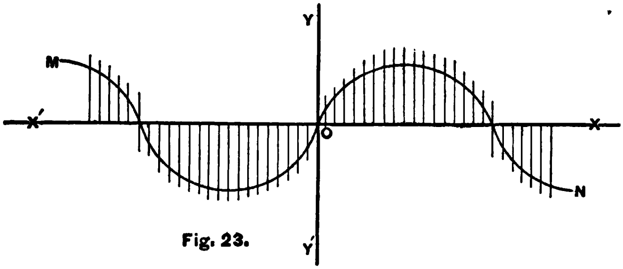


Fig. 23.

Laying down the points thus found, we have the curve MN, which is called the *Sinusoid*, or the *curve of sines.*

12. Construct and discuss

$$
\begin{aligned}
y &= \tan x; \\
y &= \cot x; \\
y &= \cos x; \\
y &= \text{vers}\ x; \\
y &= \text{covers}\ x; \\
y &= \sec x; \\
y &= \text{cosec}\ x.
\end{aligned}
$$

These loci may be constructed with sufficient accuracy without computing their numerical values. Thus, in the example,

$$y = \sec x.$$

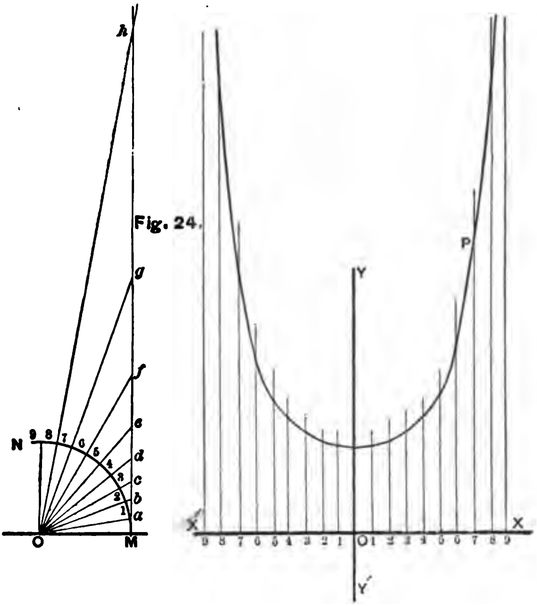


Fig. 24.

Divide a quadrant MN into any number of equal parts, say nine, so small that for practical purposes the chord and arc may be considered equal. Now measure the arcs from

M, and the secants of those arcs; then lay off the arcs on the axis of x for abscissas, and draw perpendiculars equal to the corresponding secants for the ordinates.

For example, measure O7 = the arc M7, and at 7 draw the perpendicular 7P = Oy. P will be a point of the curve.

This example may be solved in the same way as Ex. 11; thus,

When $x = \pm 0^\circ = \pm 0,$ $\quad y = 1.00.$
" $x = \pm 10^\circ = \pm .17,$ $\quad y = 1.02.$
" $x = \pm 20^\circ = \pm .35,$ $\quad y = 1.06.$
" $x = \pm 30^\circ = \pm .52,$ $\quad y = 1.16.$
" $x = \pm 40^\circ = \pm .70,$ $\quad y = 1.31.$
" $x = \pm 50 = \pm .87,$ $\quad y = 1.56.$
" $x = \pm 60^\circ = \pm 1.05,$ $\quad y = 2.00.$
" $x = \pm 70^\circ = \pm 1.22,$ $\quad y = 2.92.$
" $x = \pm 80^\circ = \pm 1.40,$ $\quad y = 5.76.$
" $x = \pm 90^\circ = \pm 1.57,$ $\quad y = \infty.$

Laying down the points thus found, we have the curve with more accuracy than by the preceding method.

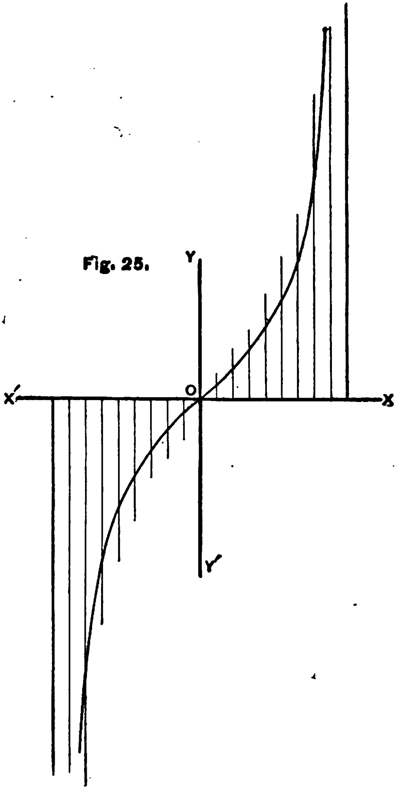
In the same way construct the equation,

$$y = \tan x.$$

When $x = 0^\circ = 0,$ $\quad y = 0.$
" $x = \pm 10^\circ = \pm .17,$ $\quad y = \pm .18.$
" $x = \pm 20^\circ = \pm .35,$ $\quad y = \pm .36.$
" $x = \pm 30^\circ = \pm .52,$ $\quad y = \pm .58.$
" $x = \pm 40^\circ = \pm .70,$ $\quad y = \pm .84.$
" $x = \pm 45^\circ = \pm .79,$ $\quad y = \pm 1.00.$
" $x = \pm 50^\circ = \pm .87,$ $\quad y = \pm 1.19.$
" $x = \pm 60^\circ = \pm 1.05,$ $\quad y = \pm 1.73.$
" $x = \pm 70^\circ = \pm 1.22,$ $\quad y = \pm 2.75.$
" $x = \pm 80^\circ = \pm 1.40,$ $\quad y = \pm 5.67.$
" $x = \pm 90^\circ = \pm 1.57,$ $\quad y = \infty.$

Laying down the points thus found, we have the curve MN.

Fig. 25.



It is clear that when we take the + values of x we must take the + values of y; and when we take the — values of x we must take the — values of y, confining ourselves to the first quadrant.

CHAPTER II.

THE RIGHT LINE.

22. I. *To find the equation of a right line, in terms of its angle with the axis of x, and its intercept on the axis of y.*

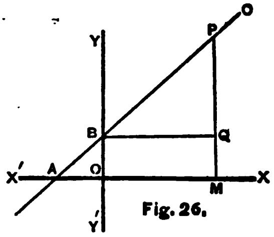


Fig. 26.

Let AC be any right line referred to the axes XX′ and YY′, and cutting the axis of y at B. Let P be any point in the given line, and draw PM perpendicular and BQ parallel to XX′; then will OM be the abscissa and MP the ordinate of the point P.

Let $\quad OM = x, \qquad MP = y, \qquad OB = b,$

$$\tan PAX = \tan PBQ = a;$$

then $\quad y = PM = PQ + QM = BQ \tan PAX + BO = ax + b,$

that is, $\qquad y = ax + b.$

But the point P is, by hypothesis, *any point* of the line AC; therefore this equation, $y = ax + b$, expresses the relation between the co-ordinates of every point of AC, and hence it is the equation of that line, by the definition in Art. 15.

OB is called the **Intercept** on the axis of y; if the line cuts the axis of y below the origin, b will be negative. a denotes the tangent of the angle which the line AC makes with the axis of x, and is positive or negative, according as the angle is $<$ or $>$ 90°.

Cor. 1.—If a is negative and b positive, the equation becomes $\qquad y = -ax + b,$

and the line cuts the axis of y *above* the origin, and makes with the axis of x an angle greater than 90°; it therefore cuts the latter at some point to the *right* of the origin, and so lies across the *first* angle.

If a and b are both positive, the equation becomes

$$y = ax + b;$$

the line lies across the *second* angle.

If a and b are both negative, the equation becomes

$$y = -ax - b;$$

the line lies across the *third* angle.

If a is positive and b negative, the equation becomes

$$y = ax - b;$$

the line lies across the *fourth* angle.

Cor. 2.—If $b = 0$, the equation becomes

$$y = ax;$$

the line passes through the origin.

If $a = 0$, the equation becomes

$$y = b;$$

the line is parallel to the axis of x.

If $a = \infty$, the line is parallel to the axis of y.

[The student may draw diagrams and verify these statements.]

Sch.—In the equation of a right line, so long as we consider the same line, a and b remain unchangeable; they are therefore called *constant quantities,* or *constants.* But x and y may have an *indefinite* number of values, since we may assign to one of them, as x, any value we please, and find the *corresponding value* of y from the equation

$$y = ax + b.$$

x and y are therefore called *variable quantities* or *variables,* as defined in Art. 15.

II. *To find the equation of a right line in terms of its intercepts on the two axes.*

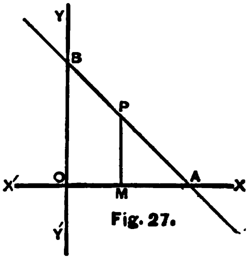


Fig. 27.

Let A and B be the points where the right line cuts the axes of x and y respectively. Let $OA = a$, $OB = b$; represent by x and y the co-ordinates OM and MP of any point P on the line. Draw PM parallel to YY'. Then, by similar triangles, we have

$$\frac{PM}{OB} = \frac{AM}{AO},$$

that is,

$$\frac{y}{b} = \frac{a - x}{a};$$

therefore,

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Cor.—By observing the signs of the arbitrary constants a and b in this equation, we can fix the position of the line with regard to the four angles, as in the preceding article.

When a and b are both positive, the line lies in the *first* angle.

When a is negative and b positive, the line lies in the *second* angle.

When a and b are both negative, the line lies in the *third* angle.

When a is positive and b negative, the line lies in the *fourth* angle.

Remark.—This form is known as the *symmetrical form* of the equation to a right line, and is frequently used. It has a close resemblance to the analogous equations of the conics; and it is applicable, as can be easily seen from the investigation, to rectangular and oblique axes alike.

III. *To find the equation of a right line in terms of the perpendicular on it from the origin, and the angle which the perpendicular makes with the axis of x.*

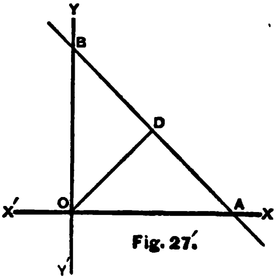


Fig. 27.

Let AB be the line; let AOD $= \alpha$, and OD $= p$. In the equation $y = ax + b$ put for a its value, $\tan \text{BAX} = -\frac{\cos \alpha}{\sin \alpha}$; and for b its value, $\text{OB} = \frac{p}{\sin \alpha}$.

Substitute these values in

$$y = ax + b,$$

and get $$y = -x\frac{\cos \alpha}{\sin \alpha} + \frac{p}{\sin \alpha},$$

or $$x \cos \alpha + y \sin \alpha = p, \tag{1}$$

which is the equation required.

Sch. 1.—The coefficients of x and y in (1) are called the **Direction Cosines** of the line, since they are the cosines of the angles which the perpendicular makes with the axes of x and y respectively. In using this form, it must be carefully remembered that α is the angle which the perpendicular makes with the *positive direction* of the axis of x, and that α may have any value from 0 to 360°, while p is always *positive*, that is, measured from O so as to bound the angle α.

Sch. 2.—This form is known as the normal form of the equation to a right line.

EXAMPLES.

1. Across which of the four angles does the line $y = -7x + 5$ lie? The line $y = 3x + 4$? The line $y = -x - 3$? The line $y = 2x - 3$?

2. Trace the line $\frac{x}{3}+\frac{y}{2}=1$; $-\frac{x}{2}-\frac{y}{3}=1$.

3. In which of the angles lie the lines $\frac{x}{2}-\frac{y}{3}=1$? $\frac{x}{3}-\frac{y}{2}=-1$? $\frac{x}{3}+\frac{y}{2}=1$? $\frac{y}{7}+x=-1$?

4. Construct the triangle, the equations of whose sides are $y=\frac{3}{4}x+3$, $y=\frac{1}{2}x-1$, $y=-\frac{4}{5}x+4$.

5. Construct the figure, the equations of whose sides are $y=x+3$, $\frac{x+y}{2}=2x-y-4\frac{1}{2}$, $y+x=3$, $x+y+3=0$.

IV. *To find the equation of a right line referred to oblique axes.*

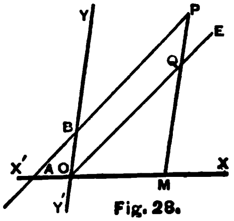


Fig. 28.

Let A and B be the points where the right line cuts the axes of x and y respectively. Draw PM parallel to YY′, and OE through the origin parallel to AB. Let x and y represent the co-ordinates OM and MP of any point P on the line. Denote the inclination of the axes by ω; and let $OB=b$, and the angle $BAX=\alpha$. Then we have,

$$y = PM = PQ + QM = OB + QM. \tag{1}$$

But
$$\frac{QM}{OM}=\frac{\sin BAO}{\sin ABO}=\frac{\sin\alpha}{\sin(\omega-\alpha)}.$$

Therefore
$$QM=\frac{\sin\alpha}{\sin(\omega-\alpha)}OM,$$

which in equation (1) gives

$$y=\frac{\sin\alpha}{\sin(\omega-\alpha)}x+b,$$

the required equation.

If we put a for $\frac{\sin\alpha}{\sin(\omega-\alpha)}$, the equation becomes

$$y=ax+b,$$

which is of the same form as the equation in Art. 22, I. The meaning of b is the same as before; a is the *ratio of the sines of the angles* which the line makes with the two axes respectively. If the axes become rectangular, $\omega = 90°$, and therefore

$$a = \frac{\sin \alpha}{\sin (90° - \alpha)} = \tan \alpha,$$

which agrees with Art. 22, I.

EXAMPLES.

1. Find the equation of a right line which makes an angle of 135° with the axis of x, and cuts off an intercept $= -3$ on the axis of y, (1) if the axes are rectangular, and (2) if they are inclined at an angle of 45°.

(1) Putting $b = -3$ and $a = \tan 135° = -1$ in the equation of Art. 22, I,

$$y = ax + b,$$

we have, for the required equation,

$$y = -x - 3.$$

(2) Putting $\alpha = 135°$, $\omega - \alpha = 45° - 135° = -90°$ in the equation of Art. 22, IV,

$$y = \frac{\sin \alpha}{\sin (\omega - \alpha)} x + b,$$

we have for the required equation,

$$y = \frac{\frac{1}{2}\sqrt{2}}{-1} x - 3,$$

or $$y = -\frac{\sqrt{2}}{2} x - 3.$$

2. Find the equation of a right line which makes an angle of 30° with the axis of x, and cuts off an intercept of 4 on the axis of y, if the axes are inclined at an angle of 60°.

Ans. $y = x + 4$.

23. *Every equation of the first degree between two variables is the equation of a right line.*

The general equation of the first degree with two variables is of the form

$$Ax + By + C = 0, \qquad (1)$$

in which A and B are the collected coefficients of x and y, and C is the sum of the absolute terms.

Solving this equation for y, we obtain,

$$y = -\frac{A}{B}x - \frac{C}{B}. \qquad (2)$$

Now we have seen in Art. 22, I, that if a right line be drawn cutting the axis of y at a distance $-\frac{C}{B}$ from the origin, and making with the axis of x an angle whose tangent is $-\frac{A}{B}$, then (2) will be the equation of this line. Hence (2), and therefore also (1) is the equation of a right line.

If $A = 0$, then (1) becomes

$$By + C = 0,$$

or

$$y = -\frac{C}{B},$$

and, from Art. 22, I, this equation represents a right line parallel to the axis of x.

If $B = 0$, then (1) becomes,

$$Ax + C = 0,$$

or

$$x = -\frac{C}{A},$$

and, by Art. 22, I, this equation represents a right line parallel to the axis of y.

If A and B have like signs, the line makes an obtuse angle with the axis of x; and if they have unlike signs, it makes an acute angle. If B and C have like signs, the line cuts the axis of y below the origin; and if they have unlike signs, it cuts the axis of y above the origin.

If $C = 0$, then (1) becomes

$$Ax + By = 0,$$

or

$$y = -\frac{A}{B}x,$$

and the line passes through the origin. Hence the equation

$$Ax + By + C = 0$$

always represents a right line.

Putting $-\frac{A}{B} = a$, and $-\frac{C}{B} = b$, and substituting in (2), we have

$$y = ax + b,$$

which is the same form as the equation in Art. 22, I.

Cor.—The equation

$$Ax + By + C = 0, \tag{1}$$

may be reduced to the form

$$x \cos \alpha + y \sin \alpha = p. \tag{2}$$

Thus, comparing (1) and (2), we see that A, B, and C are proportional to $\cos \alpha$, $\sin \alpha$, and $-p$; and therefore we have

$$\frac{A}{\cos \alpha} = \frac{B}{\sin \alpha} = -\frac{C}{p};$$

or

$$\frac{\sin \alpha}{p} = -\frac{B}{C}; \tag{3}$$

and

$$\frac{\cos \alpha}{p} = -\frac{A}{C}. \tag{4}$$

Therefore

$$\frac{\sin^2 \alpha + \cos^2 \alpha}{p^2} = \frac{B^2 + A^2}{C^2};$$

or

$$p = \frac{C}{\sqrt{A^2 + B^2}}.$$

We have taken the positive sign, because p is always positive (Art. 22, III, Sch.).

From (3) we have, $\sin\alpha = -\frac{B}{C}p = -\frac{B}{\sqrt{A^2+B^2}}$.

From (4) we have, $\cos\alpha = -\frac{A}{C}p = -\frac{A}{\sqrt{A^2+B^2}}$.

Substituting these values in (2), we have

$$-\frac{A}{\sqrt{A^2+B^2}}x - \frac{B}{\sqrt{A^2+B^2}}y = \frac{C}{\sqrt{A^2+B^2}};$$

or

$$\frac{A}{\sqrt{A^2+B^2}}x + \frac{B}{\sqrt{A^2+B^2}}y + \frac{C}{\sqrt{A^2+B^2}} = 0. \quad (5)$$

Hence any equation of the form

$$Ax + By + C = 0,$$

may be reduced to the form

$$x\cos\alpha + y\sin\alpha = p,$$

by dividing all the terms by $\sqrt{A^2+B^2}$, after changing the signs, if necessary, so that the absolute term shall be in the second member, and positive.

SCH. $-\frac{A}{\sqrt{A^2+B^2}}$ and $-\frac{B}{\sqrt{A^2+B^2}}$ are respectively the cosine and sine of the angle which the perpendicular from the origin on the line $Ax + By + C = 0$ makes with the axis of x, and $\frac{C}{\sqrt{A^2+B^2}}$ is the length of this perpendicular.

For instance, the equation $4x + 3y - 10 = 0$ may be written $\frac{4}{5}x + \frac{3}{5}y = 2$, where 2 is the perpendicular on it from the origin.

24. *To find the length of the perpendicular from any point* (x', y') *to the line* $x \cos \alpha + y \sin \alpha = p$.

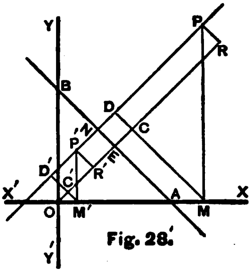


Fig. 28'.

Let (x', y') be the given point P, and AB the given line.

From the given point P draw PR parallel, and PN perpendicular to the given line AB. PN will be the perpendicular required.

From the figure we have

$$\begin{aligned} \text{PN} &= \text{PD} + \text{DN} \\ &= \text{PD} + \text{CO} - \text{EO} \\ &= \text{PM} \sin \text{DMP} + \text{OM} \cos \text{COM} - \text{EO} \\ &= x' \cos \alpha + y' \sin \alpha - p. \end{aligned}$$

We have taken P on the side of the line *opposite* the origin. If the point were taken on the *same* side as the origin, as at P', we would have,

$$\begin{aligned} \text{P'N} = \text{OE} - \text{OR}' &= \text{OE} - (\text{OC}' + \text{D'P'}) \\ &= p - x' \cos \alpha - y' \sin \alpha. \end{aligned}$$

Hence, if the equation of a line is

$$x \cos \alpha + y \sin \alpha - p = 0,$$

where p is a positive quantity, the length of the perpendicular on it from (x', y') is

$$\pm (x' \cos \alpha + y' \sin \alpha - p),$$

according as the point and the origin lie on *opposite* sides, or the *same* side of the line; that is, is equal to the result obtained by substituting in the left-hand member of the equation of the given line the co-ordinates of the given point, with the above restriction as to sign.

If the point (x', y') is *on* the line, its perpendicular becomes

$$x' \cos \alpha + y' \sin \alpha - p = 0 \quad \text{(Art. 22, III)}.$$

If the equation of the line were given in the form

$$Ax + By + C = 0,$$

we have only to reduce it to the form

$$x \cos \alpha + y \sin \alpha - p = 0 \text{ (Art. 23, Cor.)},$$

and the length of the perpendicular from any point (x', y') is

$$\pm \frac{Ax' + By' + C}{\sqrt{A^2 + B^2}}.$$

SCH.—Comparing this expression for the perpendicular from (x', y') with that for the perpendicular from the origin (Art. 23, Sch.), we see that (x', y') lies on the *same* side of the line as the origin, or on the *opposite* side, according as $Ax' + By' + C$ has the *same* sign as C, or the opposite sign.

EXAMPLE.

Find the length of the perpendicular from the origin to

$$a(x - a) + b(y - b) = 0.$$

This equation, reduced to the form

$$x \cos \alpha + y \sin \alpha - p = 0,$$

becomes (Art. 24),

$$-\frac{ax - a^2 + by - b^2}{\sqrt{a^2 + b^2}} = \sqrt{a^2 + b^2}.$$

25. *To find the equation of a right line passing through a given point.*

Let (x', y') be the given point, and the equation of the line be

$$y = ax + b. \tag{1}$$

Since the given point (x', y') is on the right line, its co-ordinates must satisfy the equation of the line; that is, the equation being true for *every* point on the line, must be true for the point (x', y'). Hence (1) becomes

$$y' = ax' + b. \tag{2}$$

Eliminating b by subtracting (2) from (1), we obtain

$$y - y' = a\,(x - x'), \tag{3}$$

which is the required equation. For it is the equation, by Art. 23, of *some* right line, since it is of the first degree between two variables; and it is the equation of a right line passing through the *given point*, because it is evidently satisfied when x' and y' are substituted in it for x and y. The constant a is the *tangent of the angle* which the line makes with the axis of x, or the *ratio of the sines of the angles* which the line makes with the two axes respectively, according as the line is referred to rectangular or oblique axes. By giving a suitable value to a, we may make equation (3) represent *any* right line which passes through the given point.

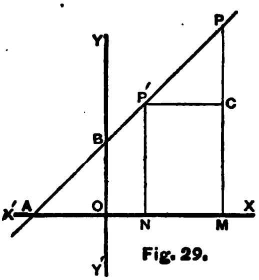


Fig. 29.

This equation (3) can easily be obtained geometrically. For let AB be any right line passing through the given point P′, the co-ordinates of which are x' and y'. Let P be *any* point on the line, x and y its co-ordinates. Draw the ordinates PM, P′N, and P′C parallel to the axis of x; then we have

$$\frac{\text{PC}}{\text{P'C}} = \tan \text{BAX} \quad \text{or} \quad = \frac{\sin \text{BAX}}{\sin \text{ABO}},$$

according as the axes are rectangular or oblique; that is,

$$\frac{y - y'}{x - x'} = \tan \text{BAX} \quad \text{or} \quad = \frac{\sin \alpha}{\sin (\omega - \alpha)},$$

according as the axes are rectangular or oblique.

Hence, $$y - y' = \tan \text{BAX}\,(x - x')$$

or $$= \frac{\sin \alpha}{\sin(\omega - \alpha)}(x - x'),$$

or $$y - y' = a\,(x - x'),$$

in which a is the *tangent of the angle* which the line makes with the axis of x, or the *ratio of the sines of the angles* which the line makes with the two axes respectively, according as the line is referred to rectangular or oblique axes. This is the same as equation (3).

26. *To find the equation of the right line which passes through two given points.*

Let the two given points be (x', y') and (x'', y''), and the equation of the line be

$$y = ax + b. \tag{1}$$

Since the two given points are on the right line, their co-ordinates must satisfy the equation of the line, giving

$$y' = ax' + b, \tag{2}$$

$$y'' = ax'' + b. \tag{3}$$

Subtracting (2) from (1), we obtain

$$y - y' = a\,(x - x'). \tag{4}$$

Subtracting (3) from (2), we obtain

$$y' - y'' = a\,(x' - x'').$$

Hence, $$a = \frac{y' - y''}{x' - x''},$$

which in (4) gives

$$y - y' = \frac{y' - y''}{x' - x''}(x - x'), \tag{5}$$

which is the required equation.

To obtain equation (5) geometrically, let P be any point (x, y) on the line AB, and P′ and P″ the two given points (x', y') and (x'', y''); then we have, from the figure,

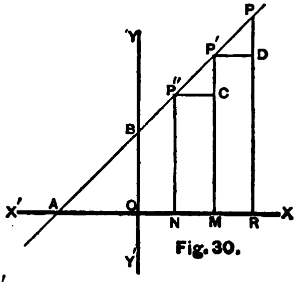


Fig. 30.

$$\frac{\text{PD}}{\text{P}'\text{D}} = \frac{\text{P}'\text{C}}{\text{P}''\text{C}},$$

or
$$\frac{y - y'}{x - x'} = \frac{y' - y''}{x' - x''}.$$

Hence,
$$y - y' = \frac{y' - y''}{x' - x''}(x - x'),$$

which is the same as equation (5).

$$\frac{y' - y''}{x' - x''} = \tan \text{BAX} = \tan \alpha,$$

if the axes are rectangular.

$$\frac{y' - y''}{x' - x''} = \frac{\sin \text{BAX}}{\sin \text{ABO}} = \frac{\sin \alpha}{\sin (\omega - \alpha)},$$

if the axes are oblique; which agrees with the results of Art. 25.

Cor. 1.—Suppose $x'' = x'$; then

$$\frac{y' - y''}{x' - x''} = \frac{y' - y''}{0} = \infty,$$

which, being the tangent of 90°, shows that the line is parallel to the axis of y, which is as it clearly should be, since, if $x'' = x'$, the points P′ and P″ are equally distant from the axis of y.

If $y'' = y'$, $\dfrac{y' - y''}{x' - x''} = \dfrac{0}{x' - x''} = 0$, which, being the tangent of 0°, shows that the line is parallel to the axis of x, which is as it clearly should be, since if $y'' = y'$, the points P′ and P″ are equally distant from the axis of x.

In the case of oblique axes, if $x'' = x'$,

$$\frac{y' - y''}{x' - x''} = \frac{y' - y''}{0} = \frac{\sin \alpha}{\sin (\omega - \alpha)},$$

therefore, $\sin (\omega - \alpha) = 0,$

and hence $\omega = \alpha$; that is, the line is parallel to the axis of y.

If $y'' = y'$, $\dfrac{y' - y''}{x' - x''} = \dfrac{0}{x' - x''} = \dfrac{\sin \alpha}{\sin (\omega - \alpha)}$; therefore $\sin \alpha = 0$, and hence the line is parallel to the axis of x.

COR. 2.—If P″ coincides with P′, we shall have,

$$x'' = x' \quad \text{and} \quad y'' = y',$$

and equation (5) becomes,

$$y - y' = \frac{0}{0}(x - x'), \tag{6}$$

which is the equation of a right line passing through a given point; and by representing the indeterminate expression $\frac{0}{0}$ by a, this equation becomes

$$y - y' = a(x - x'),$$

which agrees with equation (3), Art. 25.

COR. 3.—If we make $x' = 0$ and $y' = b$, equation (6) becomes

$$y - b = ax,$$

or

$$y = ax + b,$$

which is the equation of a line passing through a point on the axis of y, at the distance of b from the origin. This equation agrees with the one found in Art. 22, I, as it clearly should.

COR. 4.—If one of the points (x', y') be the origin, equation (5) becomes $y = \dfrac{y''}{x''}x$, which is therefore the equation of a line passing through the origin and (x'', y'').

EXAMPLES.

1. Find the equation of the right line passing through the points $(-2, 3)$ and $(3, -2)$.

Here $x' = -2$, $x'' = 3$, $y' = 3$, $y'' = -2$. Now, substituting these values in equation (5), we get

$$y - 3 = \frac{3 + 2}{-2 - 3}(x + 2),$$

and, reducing to the form $y = ax + b$, we get

$$y = -x + 1, \; \textit{Ans.}$$

2. Find the equation of the line passing through the points $(4, -2)$, $(-3, -5)$. *Ans.* $7y - 3x + 26 = 0$.

3. Find the equations of the sides of the triangle, the co-ordinates of whose vertices are $(2, 1)$, $(3, -2)$, and $(-4, -1)$.

$$\textit{Ans.} \begin{cases} x + 7y + 11 = 0, \\ 3y - x - 1 = 0, \\ 3x + 7y - 7 = 0. \end{cases}$$

4. Find the equations of the sides of the triangle, the co-ordinates of whose vertices are $(2, 3)$, $(4, -5)$, and $(-3, -6)$. *Ans.* $x - 7 = 39$, $9x - 5y = 3$, $4x + y = 11$.

5. Find the equation of the line passing through the origin and the point $(3, -2)$. *Ans.* $3y + 2x = 0$.

27. *To find the angle between two right lines whose equations are given.*

Let AC and BC be the two right lines whose equations are respectively

$$y = ax + b,$$

and $$y = a'x + b',$$

and call ϕ the angle between them.

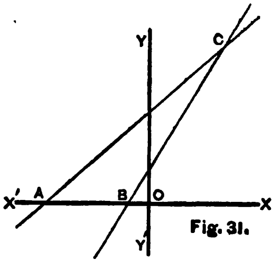


Fig. 31.

Then, Art. 22, I,

$$a = \tan \mathrm{CAX} \quad \text{and} \quad a' = \tan \mathrm{CBX},$$

and $$\tan \mathrm{ACB} = \tan (\mathrm{CBX} - \mathrm{CAX}),$$

by Trigonometry, $$= \frac{\tan \mathrm{CBX} - \tan \mathrm{CAX}}{1 + \tan \mathrm{CBX} \cdot \tan \mathrm{CAX}},$$

or $$\tan \phi = \frac{a' - a}{1 + aa'}.$$

Sch.—In applying this formula to examples, we may obtain two results numerically equal, with contrary signs. Thus, if the two lines are

$$y = 3x + 2 \quad \text{and} \quad y = 4x - 7,$$

and we let $a' = 3$ and $a = 4$, we have

$$\tan \phi = \frac{3 - 4}{1 + 12} = -\frac{1}{13}.$$

But if we let $a' = 4$, and $a = 3$, we have

$$\tan \phi = \frac{4 - 3}{1 + 12} = \frac{1}{13}.$$

This ambiguity is as it should be, since the two lines form with each other two equal acute and two equal obtuse angles; and as these angles are supplements of each other, their tangents are numerically equal, with contrary signs.

Cor. 1.—If the two lines are parallel, we have

$$\phi = 0, \quad \text{and} \quad \therefore \tan \phi = 0;$$

hence, $$\frac{a' - a}{1 + aa'} = 0,$$

which gives $$a' = a.$$

Also, if the two lines are perpendicular to each other, we have

$$\phi = 90°, \quad \text{and} \quad \therefore \tan \phi = \infty;$$

hence, $$\frac{a' - a}{1 + aa'} = \infty,$$

which gives, $$1 + aa' = 0,$$

or $$a' = -\frac{1}{a}.$$

Hence, $$y = -\frac{1}{a}x + b$$

represents a right line perpendicular to the right line

$$y = ax + b.$$

COR. 2.—We found, Art. 25, that the equation of a right line passing through a given point (x', y') is

$$y - y' = a(x - x');$$

hence, by Cor. 1, $$y - y' = -\frac{1}{a}(x - x')$$

is the equation of a line passing through a given point (x', y') and perpendicular to the line $y = ax + b$.

EXAMPLES.

1. Find the angle between the lines

$$2y + x + 1 = 0,$$
$$3y - x - 1 = 0.$$

Solving both equations with respect to y, we have

$$y = -\tfrac{1}{2}x - \tfrac{1}{2}.$$
$$y = \tfrac{1}{3}x + \tfrac{1}{3}.$$

Here $a' = -\frac{1}{2}$, $a = \frac{1}{3}$; hence,

$$\tan\phi = \frac{-\frac{1}{2} - \frac{1}{3}}{1 - \frac{1}{6}} = -1.$$

$$\therefore \quad \phi = 135^\circ.$$

2. Find the angle between the lines

$$3x + 2y - 12 = 0,$$
$$4x + y - 6 = 0.$$

Ans. $\tan\phi = \frac{5}{14}$, or $\phi = 19^\circ\,39'$.

3. Find the angle between the lines

$$y = -x + 2,$$
$$y = 3x - 6.$$

Ans. $\tan\phi = -2. \quad \therefore \phi = 116° 34'.$

4. Find the equation of the line passing through the point $(3, -4)$ and perpendicular to the line

$$5x - 4y - 52 = 0.$$

Ans. $4y = 5x - 31.$

5. Find the equation of the line passing through the point $(4, 1)$ and perpendicular to the line

$$4y = 5x - 31.$$

Ans. $5y = -4x + 21.$

28. *To find the equation of a right line which makes any given angle with a given line.*

Let ϕ be the given angle, and let $\tan\phi = m$; let the equation of the given line AB be

$$y = ax + b, \qquad (1)$$

and the equation of the required line be

$$y = a'x + b', \qquad (2)$$

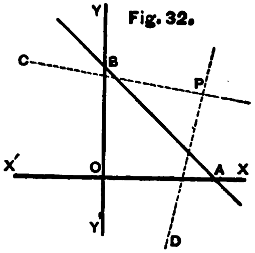


Fig. 32.

where a' is to be determined from the conditions of the problem.

Now, it is evident that the required line may be either PC or PD, since each makes the same angle with the given line AB. Hence we have, by Art. 27,

$$m = \frac{a' - a}{1 + aa'}, \quad \text{or} \quad = \frac{a - a'}{1 + aa'}.$$

Therefore,
$$a' = \frac{a \pm m}{1 \mp am},$$

which in (2) gives
$$y = \frac{a \pm m}{1 \mp am} x + b', \tag{3}$$

where b' is undetermined, as it should be, since there may be an infinite number of lines drawn fulfilling this condition, all having the same inclination to the axis of x.

COR. 1.—If the required line is to pass through a given point (x', y'), the equation will be, Art. 25,

$$y - y' = a'(x - x'),$$

or
$$y - y' = \frac{a \pm m}{1 \mp am}(x - x'). \tag{4}$$

COR. 2.—If the required line is to pass through a given point, and parallel to a given line, $m = 0$, and (4) becomes

$$y - y' = a(x - x').$$

COR. 3.—If the required line is to pass through a given point and perpendicular to a given line, $m = \infty$, and (4) becomes

$$y - y' = \frac{a \pm m}{1 \mp am}(x - x') = \frac{\frac{a}{m} \pm 1}{\frac{1}{m} \mp a}(x - x')$$

$$= \frac{\frac{a}{\infty} \pm 1}{\frac{1}{\infty} \mp a}(x - x') = \frac{\pm 1}{\mp a}(x - x'),$$

or $$y - y' = -\frac{1}{a}(x - x'),$$

which agrees with Art. 27, Cor. 2.

EXAMPLES.

1. Find the equations of the lines which pass through the point (1, 2), and make an angle of 45° with the line

$$3x + 4y + 7 = 0.$$

Here $a = -\frac{3}{4}$, $m = 1$, and taking the upper sign in equation (4), we have

$$y - 2 = \frac{-\frac{3}{4} + 1}{1 + \frac{3}{4}}(x - 1) = \frac{1}{7}(x - 1).$$

or $$7y - x - 13 = 0, \quad \textit{Ans.}$$

And taking the lower sign in (4), we have,

$$y - 2 = \frac{-\frac{3}{4} - 1}{1 - \frac{3}{4}}(x - 1) = -7(x - 1),$$

or $$y + 7x - 9 = 0, \quad \textit{Ans.}$$

2. Find the equations of the lines which pass through the point (4, 4), and make an angle of 45° with the line

$$y = 2x.$$

$$\textit{Ans.} \quad y - 4 = -3(x - 4), \text{ and}$$
$$y - 4 = \tfrac{1}{3}(x - 4).$$

3. Find the equations of the lines which pass through the origin, and make an angle of 60° with the line

$$x + y\sqrt{3} = 1.$$

$$\textit{Ans.} \quad y = \frac{1}{\sqrt{3}}x, \quad \text{and} \quad x = 0.$$

4. Find the equations of the lines which pass through the point (0, 1), and make an angle of 30° with the line

$$y + x = 2.$$

$$\textit{Ans.} \begin{cases} y - 1 = (\sqrt{3} - 2)x, & \text{and} \\ y - 1 = -(\sqrt{3} + 2)x. \end{cases}$$

5. Find the equation of the line which cuts the axis of y at a distance of 8 from the origin, and is perpendicular to the line

$$8y + 5x - 3 = 0.$$

$$\textit{Ans.} \quad 5y - 8x - 40 = 0.$$

29. *To find the co-ordinates of the point of intersection of two right lines whose equations are given.*

Let the equations of the lines be,

$$y = ax + b, \tag{1}$$

$$y = a'x + b'. \tag{2}$$

Each equation expresses a relation which must be satisfied by the co-ordinates of every point on that line; therefore, the co-ordinates of the point where the lines *intersect* must satisfy *both* equations; hence, we must make (1) and (2) simultaneous, and find the values of x and y from them. Thus,

$$x = \frac{b - b'}{a' - a}; \qquad y = \frac{ba' - b'a}{a' - a}.$$

EXAMPLES.

1. Find the co-ordinates of the intersection of the two lines

$$3x + 7y = 47,$$

$$8x - \ y = 27. \qquad \textit{Ans. } (4, 5).$$

2. Find the intersection of the lines

$$\tfrac{1}{2}x - \tfrac{1}{3}y = 1,$$

$$y = -2x + 4. \qquad \textit{Ans. } (2, 0).$$

3. Find the intersection of the two lines

$$3y + 4x - 11 = 0,$$

$$4y + 3x - 10 = 0. \qquad \textit{Ans. } (2, 1).$$

4. Find the vertices of the triangle, the equations of whose sides are

$$x + \ y = 2,$$

$$x - 3y = 4,$$

$$3x + 5y = -7.$$

$$\textit{Ans. } (-\tfrac{1}{14}, -\tfrac{19}{14}), (\tfrac{17}{2}, -\tfrac{13}{2}), (\tfrac{5}{2}, -\tfrac{1}{2}).$$

30. *Given the equations of two right lines, to find the equation of a third line passing through their point of intersection.*

The method of solving this question, which would naturally occur to the student, would be to obtain the co-ordinates of the point of intersection, by Art. 29, and then to substitute the values of these co-ordinates for x' and y' in equation (3) of Art. 25, viz., $y - y' = a(x - x')$.

The question, however, admits of an easier solution.

Let the equations of two right lines be

$$y - ax - b = 0, \tag{1}$$

$$y - a'x - b' = 0. \tag{2}$$

Multiply either equation, (2) for instance, by an arbitrary constant, k, and add the result to (1). We have,

$$(y - ax - b) + k(y - a'x - b') = 0, \tag{3}$$

which is the required equation.

For, equation (3) denotes *some* right line, since it is of the first degree (Art. 23); and it is clear that any co-ordinates which satisfy (1) and (2) must also satisfy (3), for the left member of this equation must vanish whenever $y - ax - b$ and $y - a'x - b'$ are each equal to zero. That is, equation (3) represents a line passing through a point whose co-ordinates satisfy equations (1) and (2); but this point is the intersection of the two lines, by Art. 29. Hence equation (3) denotes a line passing through the intersection of the given lines.

Since k is an arbitrary quantity, equation (3) will represent an infinite number of lines fulfilling one condition only, viz., all passing through the intersection of (1) and (2). We can therefore impose a second condition by giving the proper value to k; for example, we can make equation (3) represent a line passing through the point (x', y') by substituting x' and y' for x and y in (3), finding the value of k, and substituting this value for k in (3).

EXAMPLES.

1. Find the equation of the line passing through the intersection of

$$2x + 3y + 1 = 0, \tag{1}$$

$$3x - 4y - 5 = 0, \tag{2}$$

and the point (2, 3).

The equation of a line through the intersection of (1) and (2), by Art. 30, is

$$(2x + 3y + 1) + k(3x - 4y - 5) = 0. \tag{3}$$

As (3) is to pass through (2, 3), these co-ordinates, when substituted for x and y in (3), must satisfy it, giving us

$$(4 + 9 + 1) + k(6 - 12 - 5) = 0.$$

$$\therefore \quad k = \tfrac{14}{11},$$

which in (3) gives

$$(2x + 3y + 1) + \tfrac{14}{11}(3x - 4y - 5) = 0,$$

or,

$$64x - 23y - 59 = 0, \quad \textit{Ans.}$$

2. Given

$$\left\{\begin{aligned} 2y - x + 6 &= 0, &(1)\\ y + 4x + 8 &= 0, &(2)\\ 3y + 2x - 30 &= 0, &(3)\end{aligned}\right.$$

to find the equation of the perpendicular from the intersection of (1) and (2) to (3).

The line passing through the intersection of (1) and (2) is

$$(2y - x + 6) + k(y + 4x + 8) = 0. \tag{4}$$

Solving for y, we get

$$y = \frac{1 - 4k}{2 + k}x - \frac{6 + 8k}{2 + k}. \tag{5}$$

(5) is to be perpendicular to (3), we must have, 7, Cor. 1,

$$\frac{1-4k}{2+k} = \tfrac{3}{2}, \quad \text{or} \quad k = -\tfrac{4}{11}.$$

which in (4) gives

$$(2y - x + 6) - \tfrac{4}{11}(y + 4x + 8) = 0,$$

or, $$18y - 27x + 34 = 0, \quad \textit{Ans.}$$

3. Find the equation of the right line passing through the point (a, b), and the intersection of the right lines,

$$\frac{x}{a} + \frac{y}{b} = 1, \quad \text{and} \quad \frac{x}{b} + \frac{y}{a} = 1.$$

$$\textit{Ans.} \quad \frac{x}{a^2} - \frac{y}{b^2} = \frac{1}{a} - \frac{1}{b}.$$

4. Find the equation of the line passing through the origin and the intersection of

$$7x + 3y + 2 = 0, \quad \text{and} \quad 4x - 5y - 7 = 0,$$

$$\textit{Ans.} \quad 11y + 57x = 0.$$

31. *To find the polar equation of a right line.*

Let AB be a right line, OQ the perpendicular on it from the pole O, OX the initial line, P any point in the line. Let OQ $= p$, and the angle QOX $= \alpha$. Let (r, θ) be the polar co-ordinates of P; then

$$\text{OQ} = \text{OP} \cos \text{POQ};$$

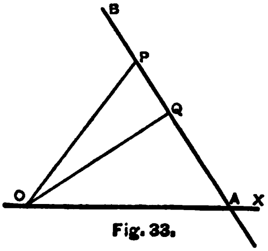


Fig. 33.

that is, $$p = r \cos(\theta - \alpha).$$

$$\therefore \quad r = \frac{p}{\cos(\theta - \alpha)}, \tag{1}$$

which is the required equation.

Cor. 1.—If the right line AB were perpendicular to the initial line, we would have $\alpha = 0$, and the equation would become

$$r = \frac{p}{\cos \theta}, \tag{2}$$

which is the equation of a right line perpendicular to the initial line.

Cor. 2.—When $\theta = 0$, (1) becomes,

$$r = \frac{p}{\cos(-\alpha)} = \frac{OQ}{\cos XOQ} = OA,$$

which is as it should be.

Cor. 3.—When $\theta = \alpha$, (1) becomes

$$r = \frac{p}{\cos 0} = p, \text{ as it should.}$$

Cor. 4.—When $\theta = 90° + \alpha$, (1) becomes

$$r = \frac{p}{0} = \infty,$$

as it should, since in this case the radius-vector becomes parallel to the line, and hence ∞.

Cor. 5.—When $\theta > 90° + \alpha$ and $< 270° + \alpha$, r is negative, as it should be, since, in order to reach the line AB, it must be produced backward from the pole, directly opposite the extremity of the arc θ measured from the initial line from the right upward to the left.

Cor. 6.—When $\theta > 270° + \alpha$ and $< 360° + \alpha$, r is positive; when $\theta = 360° + \alpha$, $r = p$, as it should; when $\theta = 360°$, $r = \dfrac{p}{\cos(-\alpha)} = OA$.

Cor. 7.—When the line AB passes through the pole,

$$r = \frac{0}{\cos(\theta - \alpha)},$$

which is 0 for every value of θ except $90° + \alpha$, for which value $r = \frac{0}{0}$, or indeterminate, as it should.

EXAMPLES.

1. Find the perpendicular distance from the point (10, 2.9) to the line

$$5y - 4x + 5 = 0. \qquad (1)$$

$$4x - 5y - 5 = 0.$$

Reducing to the normal form, we have,

$$\frac{4x}{\sqrt{4^2 + (-5)^2}} - \frac{5y}{\sqrt{4^2 + (-5)^2}} - \frac{5}{\sqrt{4^2 + (-5)^2}} = 0. \qquad (2)$$

By Art. 24, the perpendicular is

$$\frac{4(10) - 5(2.9) - 5}{\sqrt{41}} = \frac{20.5}{\sqrt{41}} = \frac{(20.5)\sqrt{41}}{41} = .5\sqrt{41}$$

$$= .5\,(6.4) = 3.2, \textit{ Ans.}$$

2. Find the intersection of the perpendicular from $(-3, 8)$ to the line $y = \frac{1}{3}x - 5$. *Ans.* $(1\frac{1}{5}, -4\frac{3}{5})$.

3. Find the angle between the lines $x + y = 1$ and $y = x + 2$; also find the co-ordinates of the point of intersection. *Ans.* 90°; $(-\frac{1}{2}, \frac{3}{2})$.

4. Find the angle between the lines $x + y\sqrt{3} = 0$ and $x - y\sqrt{3} = 2$. *Ans.* 60°.

5. Find the length of the perpendicular from the point (2, 3) to the line $2x + y - 4 = 0$. *Ans.* $\frac{3}{\sqrt{5}}$.

6. Find the lengths of the perpendiculars from each vertex to the opposite sides of the triangle (2, 1), (3, − 2), and (− 4, − 1). *Ans.* $2\sqrt{2}$, $\sqrt{10}$, $2\sqrt{10}$.

7. Find the equation of the line which passes through the point of intersection of the lines $x - 2y - a = 0$, and $x + 3y - 2a = 0$, and is parallel to the line $3x + 4y = 0$.
Ans. $3x + 4y - 5a = 0$.

8. Find the equation of the line that joins the points of intersection of the two pairs of lines,

$$\left\{\begin{array}{l} 2x + 3y - 4a = 0, \\ 2x + y - a = 0, \end{array}\right\} \text{ and } \left\{\begin{array}{l} x + 6y - 7a = 0, \\ 3x - 2y + 2a = 0. \end{array}\right\}$$

Ans. $4(x + y) - 5a = 0$.

9. The co-ordinates of two points are (3, 5) and (4, 4), respectively; find the equation of the line which bisects the distance between them and makes an angle of 45° with the axis of x. *Ans.* $y - x - 1 = 0$.

10. Find the perpendicular distance from the origin to the line $\frac{x}{2} + \frac{y}{3} = 1$. *Ans.* $\frac{6}{\sqrt{13}}$.

11. An equilateral triangle whose sides $= a$, has its vertex at the origin and its sides equally inclined to the positive directions of the axes; find the co-ordinates of the other two vertices and of the point bisecting the base.

$$\textit{Ans.} \left\{\begin{array}{ll} x = \frac{a}{4}(\sqrt{6} + \sqrt{2}), & y = \frac{a}{4}(\sqrt{6} - \sqrt{2}); \\ x = \frac{a}{4}(\sqrt{6} - \sqrt{2}), & y = \frac{a}{4}(\sqrt{6} + \sqrt{2}); \\ x = \frac{a}{4}\sqrt{6}, & y = \frac{a}{4}\sqrt{6}. \end{array}\right.$$

12. Find the equation of the lines which pass through the point (1, 3) and make an angle of 30° with the line $2y - x + 1 = 0$.

Ans. $11y - (8 \pm 5\sqrt{3})\,x - 5\,(5 \mp \sqrt{3}) = 0$.

13. Find the cosine of the angle between the lines $y - 4x + 8 = 0$ and $y - 6x + 9 = 0$. *Ans.* $\frac{25}{\sqrt{629}}$.

14. Find the equations of the diagonals of the four-sided figure, the equations of whose sides are

$$x = 4, \quad y = 5, \quad y = x, \quad y = 2x.$$

Ans. $4y = 5x$ and $3y + 2x - 20 = 0$.

15. Find the points of intersection of the lines

$$x+2y-5=0,\ \ 2x+y-7=0, \text{ and } y-x-1=0,$$

and find the area of the triangle which the lines form.

Ans. Area $= 1\frac{1}{2}$.

16. The axes of co-ordinates being inclined to each other at an angle of 45°, a right line passes through the points (2, 3) and (3, 2). Find its equation and the value of α.

Ans. $y = -x+5,\ \alpha = \tan^{-1} - (1+\sqrt{2})$.

17. The axes of co-ordinates being inclined to each other at an angle of 60°, find the equation of a line parallel to the line $(x+y=3a)$, and at a distance from it equal to $\frac{1}{2}a\sqrt{3}$.

Ans. $x+y=2a$ or $x+y=4a$ (according to the side on which the line is drawn).

18. Find the polar equation of a line the nearest point in which is 8 from the pole, and the perpendicular to which makes an angle of 30° with the initial line. Where does the line cut the initial line? What values of θ make r infinite?

Ans. $r = \dfrac{8}{\cos(\theta - 30^\circ)}$; $r = \dfrac{16}{\sqrt{3}}$; $\theta = 120^\circ$ and 300°.

19. Find the polar equation of the line perpendicular to the initial line, and which cuts it at 3 to the left of the pole. What is the value of r when $\theta = 60^\circ$? What is the value of r when θ is 120°?

Ans. $r = -\dfrac{3}{\cos\theta}$; $r = -6$; $r = +6$.

20. Find the polar co-ordinates of the intersection of the lines

$$r = \frac{2a}{\cos\left(\theta - \frac{\pi}{2}\right)} \quad \text{and} \quad r = \frac{a}{\cos\left(\theta - \frac{\pi}{6}\right)},$$

and also the angle between them.

Ans. $r = 2a,\ \theta = \frac{\pi}{2}$; angle $= \frac{\pi}{3}$.

CHAPTER III.

TRANSFORMATION OF CO-ORDINATES.

32. We saw in Art. 27 that the *general* equation of a right line is of the form $y = ax + b$, but that the equation takes simpler forms in particular cases. If the origin is *on the line,* the equation becomes $y = ax$; if the axis of x *coincides with the line,* the equation becomes $y = 0$. In a similar manner, we shall see that the equation of a *curve* often assumes simpler forms, according to the position of the origin and of the axes. For example, the circle, Fig. 33′, when referred to the axes XX′, YY′, has for its equation

$$x^2+y^2-2ax-2by+a^2+b^2-c^2 = 0,$$

but when referred to the axes xx', yy', its equation is

$$x^2 + y^2 = c^2.$$

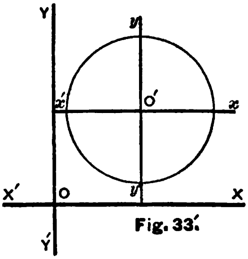


Fig. 33′.

It becomes therefore desirable to be able to change the reference of any locus from one set of axes to another, or from one system of co-ordinates to another. The operation is called the **Transformation of Co-ordinates,** and may consist either in changing the origin without disturbing the directions of the axes, or changing the directions of the axes without disturbing the origin, or changing both the position of the origin and the directions of the axes.

The axes or system from which we pass is called the **Old,** or **Primitive Axes** or **System**; the axes or system *to* which we pass is called the **New Axes** or **System.** The transformation is effected by substituting for the old co-ordinates of any point their values in terms of the new co-ordinates of the same point and certain constants.

33. *To find the formulæ for passing from one system of co-ordinates to another, the new axes being parallel to the old.*

Let OX, OY be the old axes; O$'x$, O$'y$ the new axes respectively parallel to the old. Let m and n be the co-ordinates of the new origin referred to the old axes. Let P be any point; x, y its co-ordinates referred to the old axes, and x', y' its co-ordinates referred to the new axes. Then

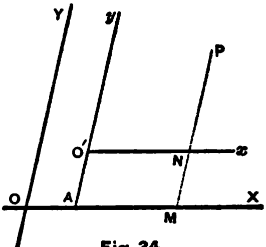


Fig. 34.

$$\text{OA} = m; \qquad \text{AO}' = n;$$

$$x = \text{OM} = \text{OA} + \text{AM} = \text{OA} + \text{O}'\text{N} = m + x';$$

$$y = \text{MP} = \text{MN} + \text{NP} = \text{AO}' + \text{NP} = n + y';$$

that is, $\qquad x = m + x', \quad \text{and} \quad y = n + y',$

which are the required formulæ.

These formulæ are equally true whether the axes be rectangular or oblique.

34. *To find the formulæ for changing the direction of the axes without changing the origin, both systems being rectangular.*

Let OX, OY be the old axes; Ox, Oy the new axes. Let the angle XO$x = \alpha$. Let P be any point; x, y its co-ordinates referred to the old axes; x', y' its co-ordinates referred to the new axes. Draw PM and PR parallel to OY and Oy respectively;

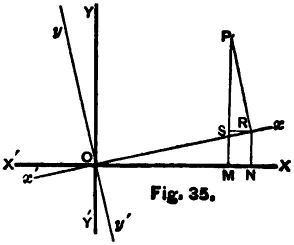


Fig. 35.

and RN and RS parallel to OY and OX respectively. Then

$$x = \text{OM} = \text{ON} - \text{SR} = \text{OR}\cos \text{XO}x - \text{PR}\sin\text{SPR}$$
$$= x'\cos\alpha - y'\sin\alpha,$$
$$y = \text{PM} = \text{RN} + \text{SP} = \text{OR}\sin\text{RON} + \text{RP}\cos\text{SPR}$$
$$= x'\sin\alpha + y'\cos\alpha,$$

which are the required formulæ.

Hence, to find what the equation of any locus becomes when referred to the new axes, we must write

$$x\cos\alpha - y\sin\alpha \quad \text{for} \quad x,$$

and

$$x\sin\alpha + y\cos\alpha \quad \text{for} \quad y.$$

35. *To find the general formulæ for passing from one rectilinear system to another.*

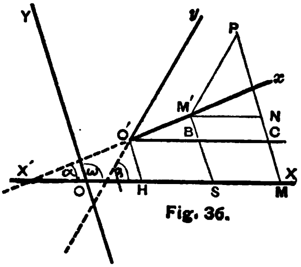


Fig. 36.

Let OX, OY be the old axes; O′x, O′y the new axes. Let the angle which the new axis of x makes with the old axis of x be α; the angle which the new axis of y makes with the old axis of x be β; the angle included between the old axes be ω; and let the co-ordinates of the new origin be OH $= m$, HO′ $= n$. Let P be any point, its co-ordinates referred to the old axes being OM $= x$, MP $= y$; its co-ordinates referred to the new axes being O′M′ $= x'$, M′P $= y'$; then we have,

$$x = \text{OM} = \text{OH} + \text{O'B} + \text{M'N}$$
$$= \text{OH} + \text{O'M'}\frac{\sin\text{O'M'B}}{\sin\text{O'BM'}} + \text{M'P}\frac{\sin\text{M'PN}}{\sin\text{M'NP}},$$

or

$$x = m + x'\frac{\sin(\omega-\alpha)}{\sin\omega} + y'\frac{\sin(\omega-\beta)}{\sin\omega}. \qquad (1)$$

$$y = \text{PM} = \text{HO}' + \text{BM}' + \text{NP}$$

$$= \text{HO}' + \text{O}'\text{M}'\frac{\sin \text{BO}'\text{M}'}{\sin \text{O}'\text{BM}'} + \text{M}'\text{P}\frac{\sin \text{NM}'\text{P}}{\sin \text{M}'\text{NP}},$$

or $$y = n + x'\frac{\sin \alpha}{\sin \omega} + y'\frac{\sin \beta}{\sin \omega}, \qquad (2)$$

which are the required formulæ.

Cor. 1.—If the old axes are rectangular, $\omega = \frac{\pi}{2}$, and (1) and (2) become

$$x = m + x' \cos \alpha + y' \cos \beta,$$
$$y = n + x' \sin \alpha + y' \sin \beta,$$

which are the *formulæ to pass from rectangular axes to oblique.*

Cor. 2.—If the new axes are rectangular, $\beta = \frac{\pi}{2} + \alpha$, and (1) and (2) become,

$$x = m + x'\frac{\sin (\omega - \alpha)}{\sin \omega} - y'\frac{\cos (\omega - \alpha)}{\sin \omega},$$

$$y = n + x'\frac{\sin \alpha}{\sin \omega} + y'\frac{\cos \alpha}{\sin \omega},$$

which are the *formulæ to pass from oblique axes to rectangular.*

Cor. 3.—If both axes are rectangular,

$$\omega = \frac{\pi}{2} \quad \text{and} \quad \beta = \frac{\pi}{2} + \alpha,$$

and (1) and (2) become

$$x = m + x' \cos \alpha - y' \sin \alpha,$$
$$y = n + x' \sin \alpha + y' \cos \alpha,$$

which are the *formulæ to pass from one set of rectangular axes to another set of rectangular axes, not parallel to the old.*

36. *To find the formulæ for passing from a rectangular to a polar system.*

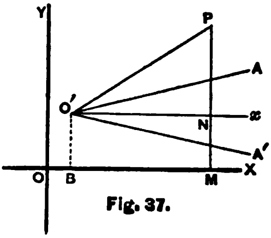


Fig. 37.

Let OX, OY be the rectangular axes; O′ the *pole*, and O′A or O′A′ the *initial line*. Let m, n, be the co-ordinates of O′ referred to the rectangular axes. Let P be any point in a locus, its co-ordinates being $OM = x$, $PM = y$, when referred to the rectangular axes; r, θ its polar co-ordinates. Let the angle $xO'A$ or $xO'A' = \alpha$. Then

$$x = OM = OB + O'N = m + O'P \cos PO'N$$
$$= m + r \cos (\theta \pm \alpha), \tag{1}$$
$$y = MP = BO' + NP = n + O'P \sin PO'N$$
$$= n + r \sin (\theta \pm \alpha), \tag{2}$$

which are the required formulæ.

Cor.—If the initial line is parallel to the old axis of x, $\alpha = 0$, and (1) and (2) become,

$$x = m + r \cos \theta, \tag{3}$$
$$y = n + r \sin \theta. \tag{4}$$

If the pole is at the origin, (3) and (4) become

$$x = r \cos \theta, \tag{5}$$
$$y = r \sin \theta. \tag{6}$$

37. *To find the formulæ for passing from a polar to a rectangular system of co-ordinates.*

It is easily seen from Fig. 37 that

$$r = \sqrt{(y-n)^2 + (x-m)^2}.$$

$$\cos (\theta \pm \alpha) = \frac{x-m}{\sqrt{(y-n)^2 + (x-m)^2}}.$$

$$\sin (\theta \pm \alpha) = \frac{y-n}{\sqrt{(y-n)^2 + (x-m)^2}}$$

38. The student will bear in mind that no change is made in the locus by any of these transformations. The assemblage of points which the new equation represents is exactly the same as that represented by the old; but the new axes to which the locus is referred occupy a different position from that occupied by the old axes; and therefore the equation which expresses this *relative* position is not the same as before.

EXAMPLES.

1. The equation of a right line is

$$3x + 5y - 15 = 0;$$

find the equation of the same line referred to parallel axes whose origin is at (1, 2). *Ans.* $3x + 5y = 2$

2. The equation of a locus is

$$x^2 + y^2 - 4x - 6y = 18;$$

what will this equation become if the origin be moved to the point (2, 3)? *Ans.* $x^2 + y^2 = 31$.

3. The equation of a locus is $y^2 - x^2 = 16$; what will this equation become if transformed to axes bisecting the angles between the given axes? *Ans.* $xy = 3$.

4. Transform the equation $2x^2 - 5xy + 2y^2 = 4$ from axes inclined to each other at an angle of 60°, to the axes which bisect the angles between the given axes.
Ans. $x^2 - 27y^2 + 12 = 0$.

5. Transform the equation $y^2 + 4ay \cot \alpha - 4ax = 0$ from a rectangular system to an oblique system inclined at an angle α, the origin remaining the same, and the new axis of x coinciding with the old. *Ans.* $y^2 \sin^2 \alpha = 4ax$.

6. The equation of a locus is $x^4 + y^4 + 6x^2y^2 = 2$; what will be the equation if the axes are turned through an angle of 45°? *Ans.* $x^4 + y^4 = 1$.

7. Transform $x^2 + y^2 = 7ax$ to polar co-ordinates, the pole being at the origin, and the initial line coincident with the axis of x. *Ans.* $r = 7a \cos \theta$.

8. Change the equations $r^2 = a^2 \cos 2\theta$ and $r^2 \cos 2\theta = a^2$ into equations between x and y.

Ans. $x^2 + y^2 = a^2 (x^2 - y^2)$ and $x^2 - y^2 = a^2$.

39. The following exercises are designed to give the student an opportunity for making an effort to produce the equations himself. The fundamental idea of Analytic Geometry is that every geometric condition to be fulfilled by a point leads to an equation which must be satisfied by its co-ordinates. It is important that the student should become able to express by an equation any given geometric condition; he should understand that ability to investigate, to reason for himself, is the chief object for the attainment of which he should strive. For this purpose he should diligently apply himself in working out examples, until he has acquired readiness and accuracy in so doing. In attempting to solve these examples, the student will find that very much depends upon a proper selection of the origin and axes of co-ordinates, and the application of the proper equations and formulæ. He should, in every case, consider the problem well, and form a definite plan before he attempts the solution. He will often be unable to carry out his original plan, and will have to abandon it, although it may have seemed at first the most suitable. Such failures, however, are not to be considered as waste of time; for it is only by thorough application that the student is enabled, gradually, to become expert in obtaining solutions; and a *failure* will often suggest some method by which a problem may be solved.

[The student need not necessarily tarry till he has mastered all the examples in any one article.]

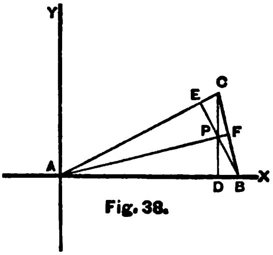


Fig. 38.

1. Prove that the perpendiculars drawn from the vertices of a triangle to the opposite sides meet in a point.

Let ABC be the triangle; AF, BE, CD the perpendiculars. Assume AX and AY as the rectangular axes; and let the co-ordinates of B and C be x'', 0, and x', y', respectively. Now, if it can be shown that x' is the abscissa of the point of intersection of the perpendiculars AF and BE, the proposition will be proved. We therefore have to find the equations of AF and BE, and then their intersection.

Since AC passes through the origin and the point C, (x', y'), its equation (Art. 26, Cor. 4) is

$$y = \frac{y'}{x'} x. \tag{1}$$

Since BC passes through B (x'', 0) and C (x', y'), its equation (Art. 26) is

$$y = \frac{y'}{x' - x''} (x - x''). \tag{2}$$

Since BE passes through B (x'', 0) and is perpendicular to (1), its equation (Art. 28, Cor. 3) is

$$y = -\frac{x'}{y'} (x - x''). \tag{3}$$

Since AF passes through the origin (0, 0) and is perpendicular to (2), its equation is

$$y = -\frac{x' - x''}{y'} x. \tag{4}$$

At the point P, where (3) and (4) intersect, their ordinates must be identical; hence, equating their values, we have

$$\frac{x'}{y'}(x-x'') = \frac{x'-x''}{y'}x.$$

$$\therefore \quad x = x'.$$

That is, the abscissa of the point of intersection of AF and BE is the same as the abscissa of the point C; therefore the perpendicular CD passes through the intersection P. [This solution is similar to the one given by Puckle in his Conic Sections, p. 77.]

2. Given the base $(=2m)$ of a triangle, and the difference between the squares of its sides $(=n^2)$, to find the locus of its vertex.

Take for axes the base and a perpendicular through its middle point, and let the co-ordinates of the vertex C be x, y. Then

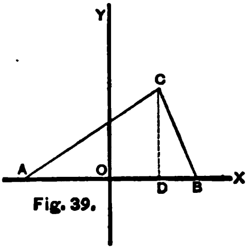


Fig. 39.

$$\overline{AC}^2 = (m+x)^2 + y^2;$$

$$\overline{BC}^2 = (m-x)^2 + y^2.$$

$$\therefore \quad \overline{AC}^2 - \overline{BC}^2 = 4mx = n^2,$$

or

$$x = \frac{n^2}{4m},$$

the equation required. The locus is therefore a line perpendicular to the base, at the distance of $\frac{n^2}{4m}$ from the middle point.

3. A line is drawn parallel to the base of a triangle, and its extremities are joined transversely to those of the base; to find the locus of the point of intersection of the joining lines.*

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* This solution is from Salmon's Conic Sections, p. 44.

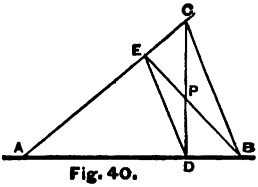

Fig. 40.

Take for axes the sides of the triangle, AB and AC. Let AB $= a$, AC $= b$, and let the lengths of the proportional intercepts made by the parallel be ka, kb. Then the equations of the transversals will be as follows:

Equation of BE (Art. 22) is $\frac{x}{a} + \frac{y}{kb} = 1.$

Equation of CD (Art. 22) is $\frac{x}{ka} + \frac{y}{b} = 1.$

Subtract one from the other; divide by the constant, $\left(1 - \frac{1}{k}\right)$, and we get for the equation of the locus,

$$\frac{x}{a} - \frac{y}{b} = 0, \qquad \text{or} \qquad y = \frac{b}{a}x,$$

a right line passing through the origin and the middle of BC.

4. Given the base of a triangle $= 2m$, and the sum of the cotangents of the base angles $= n$, to find the locus of its vertex.

From Fig. 39 we have,

$$\cot A = \frac{AD}{DC} = \frac{m + x}{y},$$

$$\cot B = \frac{m - x}{y}.$$

Hence the required equation is

$$\frac{2m}{y} = n, \qquad \text{or} \qquad y = \frac{2m}{n},$$

a right line parallel to the base, at the distance $\frac{2m}{n}$ from it.

5. Given the base of a triangle $= 2m$, and the sum of the sides $= s$; let the perpendicular to the base be produced beyond the vertex until its whole length is equal to one of the sides; to find the locus of the extremity of the perpendicular.

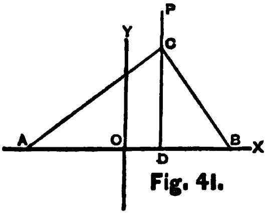

Fig. 41.

Take the origin at the middle of the base, axes rectangular, as in Fig. 41. The abscissa of P is $\mathrm{OD} = x$, and the ordinate is $\mathrm{DP} = \mathrm{AC} = y$.

$$\mathrm{BC} = s - \mathrm{AC} = s - y;$$

$$\overline{\mathrm{BC}}^2 = \overline{\mathrm{AC}}^2 + \overline{\mathrm{AB}}^2 - 2\mathrm{AB} \times \mathrm{AD},$$

or $$(s-y)^2 = y^2 + 4m^2 - 4(m+x)m,$$

or $$s^2 - 2sy = -4mx;$$

therefore, $$y = \frac{2m}{s}x + \frac{s}{2},$$

which is the equation of the required locus, the equation of a right line.

6. Prove that the three perpendiculars through the middle points of the sides of a triangle meet in a point.

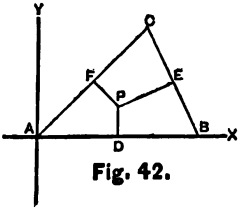

Fig. 42.

Suggestions.—1st, find equation of AC; 2d, find equation of BC; 3d, find equation of FP perpendicular to AC; 4th, find equation of EP perpendicular to BC; 5th, find abscissa of point of intersection of FP and EP; ∴ etc.

7. Prove that the lines drawn from the vertices of a triangle to the middle points of the opposite sides pass through the same point.

[Take for axes EB and EC in Fig. 43.]

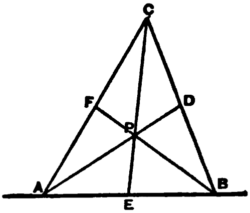

Fig. 43.

8. Given two fixed points A and B, one on each of the axes of co-ordinates, at the respective distances a and b from the origin; if A′ and B′ be taken on the axes so that $OA' + OB' = OA + OB$, find the locus of the intersection of AB′ and A′B. *Ans.* $x + y = a + b$.

9. $PP' = a$, and $QQ' = b$ are any two parallels to the sides of a given parallelogram, to find the locus of the intersection of the lines PQ and P′Q′.

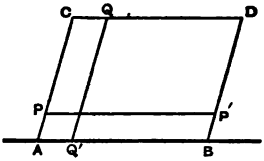

Fig. 44.

Take AB, AC for the axes of co-ordinates; let $AQ' = m$, $AP = n$. Then, 1st, find the equation of the line joining P $(0, n)$ to Q (m, b); 2d, find the equation of the line joining P′ (a, n) to Q′ $(m, 0)$; 3d, add these two equations together, and get for the locus, $y = \frac{b}{a}x$, the equation of the diagonal of the parallelogram.

10. On the two sides of a right-angled triangle, squares are constructed; from the acute angles, diagonals are drawn, crossing the triangle to the vertices of these squares; and from the right angle a perpendicular is let fall upon the hypothenuse; prove that the diagonals and the perpendicular meet in one point. [Take the two sides for axes, and call their lengths a and b.]

CHAPTER IV.

THE CIRCLE.

40. We shall now consider loci whose equations are of the second degree, beginning with the circle, which is the simplest of these loci.

A circle is a plane figure bounded by a line every point of which is equally distant from a point within called the centre. In Analytic Geometry, the term **Circle** is applied indifferently both to the area of the figure and to the bounding line, while in Plane Geometry the term is confined to the area, the bounding line being called the circumference.

41. *To find the equation of the circle whose centre and radius are given.*

Let C be the centre of the circle, P any point on its circumference, and r the radius of the circle. Let a, b be the co-ordinates of C; x, y the co-ordinates of P. Draw CN, PM parallel to OY, and CB parallel to OX. Then we have

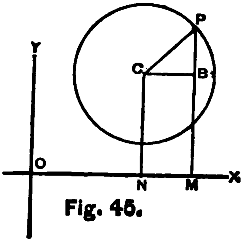

Fig. 45.

$$\overline{CB}^2 + \overline{BP}^2 = \overline{CP}^2;$$

or
$$(x - a)^2 + (y - b)^2 = r^2. \quad (1)$$

This equation is true for every position of P; hence it expresses the relation between the co-ordinates of every point of the circle, and is therefore the required equation.

If the axes are oblique, and inclined to each other at an angle $= \omega$, the equation is

$$(x - a)^2 + (y - b)^2 + 2(x - a)(y - b)\cos\omega = r^2. \quad (2)$$

COR. 1.—If the origin be transferred to the centre of the circle, then $a = 0$, $b = 0$, and equation (1) becomes

$$x^2 + y^2 = r^2. \tag{3}$$

This equation may be written

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1, \tag{4}$$

a form analogous to that of the equation of a right line,

$$\frac{x}{a} + \frac{y}{b} = 1. \quad \text{[See Art. 22, II.]}$$

COR. 2.—If the origin be transferred to the circumference, and the diameter which passes through the origin be taken for the axis of x, then $a = r$, $b = 0$, and equation (1) becomes

$$(x - r)^2 + y^2 = r^2,$$

or

$$y^2 = 2rx - x^2. \tag{5}$$

It may be observed here that, if the origin is on the curve, there will be no term which does not involve either x or y; for the equation is satisfied by the co-ordinates of the origin, $x = 0$, $y = 0$. The same argument proves that *if an equation of any degree wants the absolute term, the curve represented passes through the origin.*

In equation (5) we suppose the origin to be at the *left-hand* vertex of the diameter. This convention is adopted by custom.

COR. 3.—To find where (1) cuts the axis of x, we make $y = 0$, and have

$$x = a \pm \sqrt{r^2 - b^2}.$$

If $b^2 < r^2$, the two values of x are *real* and *unequal*, showing that the curve cuts the axis of x in two points.

If $b^2 = r^2$, the two values of x are *real* and *equal*, showing that the curve touches the axis of x; that is, is tangent to it.

If $b^2 > r^2$, the two values of x are *imaginary*, showing that the curve does not cut the axis of x.

Similarly, it may be shown that the curve cuts the axis of y in two points, is tangent to the axis of y, or does not cut the axis of y, according as $a^2 < r^2$, $= r^2$, or $> r^2$.

COR. 4.—To find where (3) cuts the axis of x, we make $y = 0$, and get $x = \pm r$, showing that the curve cuts the axis of x in two points on different sides of the origin, at the distance r from it.

To find where (3) cuts the axis of y, we make $x = 0$, and obtain $y = \pm r$, showing that the curve cuts the axis of y at r above and r below the origin.

Solving (3) with respect to y, we obtain,

$$y = \pm \sqrt{r^2 - x^2},$$

which shows that, for every value of x between $+r$ and $-r$, y has two real values, numerically equal, with contrary signs; hence the curve is symmetrically situated with respect to the axis of x. If $x = +r$ or $-r$, the two values of y are equal to 0, which shows that the ordinates at these two points are tangent to the curve. If $x > +r$ or $< -r$, y becomes imaginary, which shows that the curve does not extend beyond the two tangents just described.

Similarly it may be shown that the curve is symmetrical with respect to the axis of y, and that it does not extend beyond the two tangents drawn through the extremities of the vertical diameter.

COR. 5.—To find where (5) cuts the axis of x, we make $y = 0$, and obtain

$$x(2r - x) = 0.$$

This equation is satisfied by supposing $x = 0$, or

$$2r - x = 0,$$

from the last of which we get,

$$x = 2r;$$

hence the curve cuts the axis of x at the origin, and at the distance $2r$ to the right of it.

To find where the curve cuts the axis of y, we make $x = 0$, and obtain $y = \pm 0$, which shows that the curve touches the axis of y at the origin.

COR. 6.—If (x', y') and (x'', y'') be any two points on the curve, we shall have from (3),

$$y'^2 = r^2 - x'^2; \qquad y''^2 = r^2 - x''^2.$$

Hence, forming a proportion, we have,

$$y'^2 : y''^2 :: (r - x')(r + x') : (r - x'')(r + x'').$$

That is, the squares of any two ordinates to any diameter are to each other as the rectangles of the segments into which they divide the diameter.

EXAMPLES.

1. The equation of a circle is

$$x^2 + y^2 + 4x - 8y - 5 = 0;$$

find the co-ordinates of the centre, and the radius.

Writing the equation in the form of (1), it becomes

$$(x + 2)^2 + (y - 4)^2 = 25;$$

from which we see that the co-ordinates of the centre and the radius are $(-2, 4)$ and 5.

2. The equations of two circles are

$$x^2 + y^2 - 2x + 4y + 1 = 0;$$

$$3x^2 + 3y^2 - 5x - 7y + 1 = 0;$$

find the co-ordinates of the centre, and the radius in each circle.

Ans. $\begin{cases} (1, -2) \text{ and } 2 \text{ in the first case;} \\ (\frac{5}{6}, \frac{7}{6}) \text{ and } \frac{1}{6}\sqrt{62} \text{ in the second.} \end{cases}$

3. Form the equation of the circle whose centre is (3, 4), and whose radius $= 2$.

Ans. $x^2 + y^2 - 6x - 8y + 21 = 0.$

4. Form the equation of the circle whose centre is $(5, -3)$, and whose radius $= \sqrt{7}$, when $\omega = 60^\circ$.

Ans. $x^2 + y^2 + xy - 7x + y + 12 = 0.$

5. Find the equation of the circle which passes through the points $(-6, -1)$, $(0, 0)$, $(0, -1)$; and also the co-ordinates of the centre, and the radius.

[These three sets of co-ordinates must each satisfy equation (1), giving three equations from which to obtain the values of a, b, and r.]

Ans. $x^2 + y^2 + 6x + y = 0$; and $(-3, -\frac{1}{2})$ and $\frac{1}{2}\sqrt{37}$.

6. Find the equation of a circle referred to its diameter and left-hand vertex that shall pass through the point (2, 3).

Ans. $y^2 = \frac{13}{2}x - x^2.$

42. *To find the equation of the tangent at any point of a circle.*

The **Tangent** *to any curve is the line joining two indefinitely near points on that curve.*

Hence, its equation will be found by first forming the equation of the secant drawn through any two points (x', y'), (x'', y'') on the curve, and then allowing the first point to remain fixed while the second moves on the curve up to the first; the secant in its limiting position will become the tangent to the curve at the first point, and the equation of the secant will become the equation of the tangent.

The equation of the circle, the origin at the centre, is

$$x^2 + y^2 = r^2. \tag{1}$$

The equation of the secant through (x', y') and (x'', y'') is

$$y - y' = \frac{y' - y''}{x' - x''}(x - x'). \tag{2}$$

Since (x', y') and (x'', y'') are both on the circle, they ill satisfy equation (1) ; therefore,

$$x'^2 + y'^2 = r^2, \tag{3}$$

and

$$x''^2 + y''^2 = r^2. \tag{4}$$

Subtract (4) from (3), transpose and factor, and we have $(y' - y'')(y' + y'') = -(x' - x'')(x' + x'')$; from which we obtain $\frac{y' - y''}{x' - x''} = -\frac{x' + x''}{y' + y''}$. Hence, substituting in (2), it becomes

$$y - y' = -\frac{x' + x''}{y' + y''}(x - x'). \tag{5}$$

Now when the second point coincides with the first, we have $x'' = x'$, $y'' = y'$; therefore (5) becomes

$$y - y' = -\frac{x'}{y'}(x - x'), \tag{6}$$

which is the equation of the tangent at the point (x', y'), $-\frac{x'}{y'}$ being the tangent of the angle which the tangent to the curve at the point (x', y') makes with the axis of x.

Multiplying (6) by y', transposing, and remembering that $x'^2 + y'^2 = r^2$, we get

$$xx' + yy' = r^2, \tag{7}$$

a form very similar to the equation of the circle.

Equation (7) may be written,

$$\frac{xx'}{r^2} + \frac{yy'}{r^2} = 1, \tag{8}$$

a form analogous to that of the equation of a right line. (See Art. 22, II.)

43. *To find the equation of the normal at any point of a circle.*

The **Normal** *at any point of a curve is the right line drawn through that point at right angles to the tangent to the curve at the same point.*

The equation of the tangent to a circle at the point (x', y'), Art. 42, is

$$xx' + yy' = r^2,$$

or

$$y = -\frac{x'}{y'}x + \frac{r^2}{y'};$$

therefore (Art. 28, Cor. 3), the equation of a right line through (x', y') perpendicular to the tangent at the same point, is

$$y - y' = \frac{y'}{x'}(x - x'), \qquad \text{or} \qquad y = \frac{y'}{x'}x.$$

Since this equation is satisfied by the values $x = 0$, $y = 0$, the normal at any point passes through the origin of co-ordinates, that is, through the centre of the circle.

Sch. 1.—The **Subtangent** is the distance from the point in which the tangent intersects the axis of x to the foot of the ordinate from the point of tangency; or it is the projection of the corresponding portion of the tangent upon the axis of x.

Sch. 2.—The **Subnormal** is the distance from the foot of the ordinate of the point in the curve to which the normal is drawn to the point of intersection of the normal with the axis of x; or it is the projection of the corresponding portion of the normal upon the axis of x.

In Fig. 46, TP is the *tangent* to the curve at the point P; MP is the *ordinate of the point of tangency;* PN, the *normal;* MT, the *subtangent;* MN, the *subnormal.*

From the figure we have

$$\mathrm{TM} = \frac{\mathrm{MP}}{\tan \mathrm{MTP}};$$

or, *Subtangent* (Art. 43) $= -\dfrac{y'^2}{x'}$

Also, $\mathrm{MN} = \mathrm{MP} \tan \mathrm{MNP}$;

or, *Subnormal* $= -x'$,

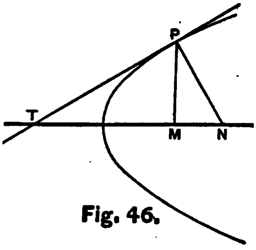

Fig. 46.

which shows that the normal passes through the centre of the circle. (See Art. 43.)

EXAMPLES.

1. Find the equation of the tangent to the circle

$$x^2 + y^2 = 25,$$

at the point whose abscissa is $\sqrt{7}$.

Ans. $\sqrt{7}x \pm 3y = 25$.

2. Find the subtangent in the last example.

Ans. Subtangent $= -\dfrac{9}{\sqrt{7}}$.

3. Find the equations of two right lines which touch the circle $x^2 + y^2 = 10$, at points the common abscissa of which is one. *Ans.* $x \pm 3y = 10$.

44. If the equation of a circle be given in the form

$$(x - a)^2 + (y - b)^2 = r^2, \tag{1}$$

we may find the equation of the tangent at any point, in the same way as in Art. 42.

Let (x', y') be the point on the circle at which the tangent is drawn; (x'', y'') a second point on the circle; then these points will satisfy (1), giving

$$(x' - a)^2 + (y' - b)^2 = r^2, \tag{2}$$

$$(x'' - a)^2 + (y'' - b)^2 = r^2. \tag{3}$$

Subtract (3) from (2), transpose and factor, and we have

$$(x'-x'')(x''+x'-2a)+(y'-y'')(y''+y'-2b)=0,$$

from which we obtain,

$$\frac{y'-y''}{x'-x''}=-\frac{x''+x'-2a}{y''+y'-2b}. \tag{4}$$

Substituting (4) in the equation of the secant through (x', y') and (x'', y''), we have

$$y-y'=-\frac{x''+x'-2a}{y''+y'-2b}(x-x'). \tag{5}$$

When the second point coincides with the first, we have $x''=x'$, $y''=y'$, and (5) becomes

$$y-y'=-\frac{x'-a}{y'-b}(x-x').$$

Clearing of fractions, transposing, and factoring, we have

$$(x-a)(x'-a)+(y-b)(y'-b)=r^2, \tag{6}$$

which is the equation of the tangent required, a form easily remembered, from its similarity to the corresponding equation of the circle.

EXAMPLES.

1. Find the equation of the tangent to the circle $(x-2)^2+(y-3)^2=10$, at the point (5, 4).

Ans. $3x+y=19$.

2. Find the equation of the tangent to the circle $x^2+y^2-2y-3x=0$, at the origin.

Ans. $2y+3x=0$.

3. Find the equation of the right line passing through the origin, and tangent to the circle $x^2+y^2-3x+4y=0$.

Ans. $4y-3x=0$.

4. Find the equations of the tangents from the origin to $x^2+y^2-6x-2y+8=0$.

Ans. $x-y=0$, and $x+7y=0$.

45. *To find the co-ordinates of the points in which a given right line* $y = ax + b$ *intersects a given circle* $x^2 + y^2 = r^2$.

Equating to each other the two values of y found from the two equations, we have, for determining the abscissas of the points of intersection, the equation

$$(1 + a^2)x^2 + 2abx = r^2 - b^2;$$

hence,
$$x = \frac{-ab \pm \sqrt{(1 + a^2)r^2 - b^2}}{1 + a^2},$$

giving us two roots, *real* and *unequal*, *equal* or *imaginary*, according as $(1 + a^2)r^2$ is greater than, equal to, or less than b^2.

Hence, when the first of these conditions occurs, the right line will meet the circle in two *real* and *different* points; when the second, in two *consecutive* or *coincident* points, becoming a tangent (see Art. 42); when the third, in two *imaginary* points.

By **Consecutive Points** is meant *points whose distance apart is infinitely small;* that is, so small that we cannot assign a value too small for it. We may assign the value 0, and take the points as absolutely coincident, and hence they may be designated as **Coincident Points**, which is the language of pure Geometry; the term *consecutive* is peculiar to the Analytic method.

By an **Imaginary Point** is meant a point, one or both of whose co-ordinates are imaginary. It really means, when translated into the language of pure Geometry, that *the corresponding point not only does not exist, but is impossible.* It is a purely analytic conception, and has no geometric meaning, just as imaginary roots of an equation have no arithmetic meaning attached to them; and yet there are cases in which the line joining two imaginary points is *real,* and has all the geometric properties of the corresponding line joining two real points.

EXAMPLES.

1. Find the points of intersection of the circle $x^2+y^2=25$, and the line $y+x+1=0$.

Ans. $(-4, 3)$ and $(3, -4)$.

2. Find the points of intersection of the circle $x^2+y^2=25$, and the line $3y+4x+25=0$.

Ans. The line touches the circle at $(-4, -3)$.

3. Find the intersections of $x^2+y^2=65$ and $3x+y=25$.

Ans. $(7, 4)$ and $(8, 1)$.

4. Find the intersections of $x^2+y^2=25$ and $x+y=-5$.

Ans. $(0, -5)$ and $(-5, 0)$.

5. Find the points in which the circle $x^2+y^2=9$ intersects the lines

$$x+y+1=0,$$
$$x+y-1=0,$$
$$2x+y\sqrt{5}=9.$$

$$\textit{Ans.}\begin{cases}(1.55, \ -2.55) \text{ and } (-2.55, \ 1.55)\,; \\ (2.55, \ -1.55) \text{ and } (-1.55, \ 2.55)\,; \\ \text{tangent at } (2, \ \sqrt{5}).\end{cases}$$

46. *To find the length of the tangent drawn from any point to the circle.*

$$(x-a)^2+(y-b)^2=r^2. \qquad (1)$$

Let (x', y') be any point in the plane of the circle whose centre is (a, b); then (Art. 9), for the distance between (x, y) and (a, b) we have

$$\sqrt{(x'-a)^2+(y'-b)^2};$$

and since this distance is the hypothenuse of a right-angled triangle whose two sides are the radius of the circle and the corresponding tangent, we have, calling the tangent t,

$$t^2=(x'-a)^2+(y'-b)^2-r^2; \qquad (2)$$

from which we see that if the co-ordinates of any point be substituted for x and y in the equation of a circle, the result will be the square of the length of the tangent drawn from that point to the circle.

EXAMPLES.

1. Given the two circles

$$(x-2)^2+(y-\tfrac{5}{2})^2=3\tfrac{1}{4};$$
$$(x+3)^2+(y+4)^2=34;$$

to find the locus of a point such that the tangents drawn from it to the two circles shall be of equal length.

Let (x, y) be the point; then by the present Art., the squares of the tangents from (x, y) to the two circles are

$$(x-2)^2+(y-\tfrac{5}{2})^2-3\tfrac{1}{4}, \tag{1}$$
$$(x+3)^2+(y+4)^2-34. \tag{2}$$

Now since these two tangents are to be of equal length, (1) must equal (2); from which we find,

$$10x+13y-16=0, \tag{3}$$

which is the equation of the required locus; this locus is a right line, and is called *the radical axis of the two given circles.* Hence, *the* **Radical Axis of two Circles** *is a right line, from any point of which if two tangents be drawn to both circles, the two tangents will be of equal lengths.*

The equation of the line through the centres of the given circles is

$$y-\tfrac{5}{2}=\left(\frac{\frac{5}{2}+4}{2+3}\right)(x-2);$$

or $$10y-13x+1=0. \tag{4}$$

Comparing (3) and (4) we see (Art. 27, Cor. 1) that the radical axis of the circles is perpendicular to the line which joins their centres.

2. Find the radical axis of

$$(x-5)^2+(y-4)^2=4,$$
$$(x-2)^2+(y-1)^2=1.$$

Ans. $x+y=\frac{11}{2}$.

3. Find the radical axis of

$$(x-1)^2+(y-2)^2=6,$$
$$(x-2)^2+(y-3)^2=8.$$

Ans. $x+y=3$.

4. Find a point from which the tangents drawn to the three circles,

$$(x-1)^2+(y-2)^2=7, \qquad (1)$$
$$(x-3)^2+y^2=5, \qquad (2)$$
$$(x+4)^2+(y+1)^2=9, \qquad (3)$$

shall be equal.

It is clear that the point lies on the radical axis of (1) and (2), and on the radical axis of (1) and (3); it is therefore at their intersection. This point is called the **Radical Centre.**

5. Find the radical centre of $(x-5)^2+(y-6)^2=4$, $(x-3)^2+(y-1)^2=1$, $(x+1)^2+(y+2)^2=9$.

Ans. $(-\frac{79}{28}, \frac{166}{28})$.

47. *Tangents are drawn to a circle from a given external point; to find the equation of the chord of contact.*

Let x', y' be the co-ordinates of the external point P′; x'', y'' the co-ordinates of the point P″, where one of the tangents from P′ meets the circle; x_1, y_1 the co-ordinates of the point P_1, where the other tangent from P′ meets the circle. Then $P''P_1$ will be the line whose equation is required.

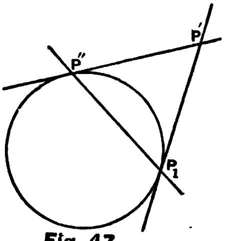

Fig. 47.

The equation of the tangent at P″ (see Art. 42) is

$$xx'' + yy'' = r^2, \tag{1}$$

and since this tangent passes through P′, the co-ordinates of P′ must satisfy the equation; hence we have

$$x'x'' + y'y'' = r^2. \tag{2}$$

Also the equation of the tangent at P_1 is

$$xx_1 + yy_1 = r^2, \tag{3}$$

and since this tangent passes through P′, its co-ordinates must satisfy the equation, giving

$$x'x_1 + y'y_1 = r^2. \tag{4}$$

Hence it follows that the equation of the chord of contact, $P''P_1$, is

$$xx' + yy' = r^2. \tag{5}$$

For (5) is the equation of *some right* line (Art. 23); and this line passes through P″, for (5) is satisfied by the values $x = x''$, $y = y''$, as we see from (2). Also, the line passes through P_1, for (5) is satisfied by the values $x = x_1$, $y = y_1$, as we see from (4). Hence (5) is the required equation.

48. *Through any fixed point a chord is drawn to a circle, and tangents to the circle are drawn at the extremities of the chord; to find the equation of the locus of the intersection of these tangents when the chord is turned about the fixed point.*

Let (x', y') be the fixed point P′ through which the chord passes; and (x'', y'') the point P″ in which the two tangents drawn at the extremities Q, R of the chord, intersect. The equation of the chord of contact (Art. 47), is

$$xx'' + yy'' = r^2. \tag{1}$$

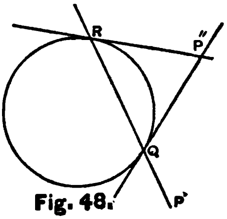

Fig. 48.

But since this chord passes through (x', y'), we have

$$x'x'' + y'y'' = r^2. \tag{2}$$

Now if the chord turns about the fixed point (x', y'), the intersection (x'', y'') of the two corresponding tangents is *any* point in the required locus; that is, the co-ordinates x'', y'' of any point in the required locus satisfy the equation

$$xx' + yy' = r^2, \tag{3}$$

which is therefore the equation required, and the locus is a right line.

49. The line $xx' + yy' = r^2$ is called the **Polar** of the point (x', y') with regard to the circle $x^2 + y^2 = r^2$; and the point (x', y') is called the **Pole** of the line.

It will be seen (Art. 48) that if (x', y') be *any point whatever*, the equation $xx' + yy' = r^2$ represents the locus of the intersection of the tangents at the extremities of the chord through (x', y').

If (x', y') be an *external point*, the equation $xx' + yy' = r^2$ represents the chord of contact (Art. 47).

If (x', y') be *on the circle*, the equation $xx' + yy' = r^2$ represents the tangent at that point (Art. 42).

That is, the three equations are identical; the position of the point (x', y') in Art. 48 is not subject to any limitation; hence, wherever the point (x', y') may be, the equation $xx' + yy' = r^2$ represents the locus of the intersection of tangents drawn at the extremities of chords which all pass through (x', y'). If the point be *without* the circle, this locus is identical (Art. 47) with the chord joining the points of contact of tangents drawn from (x', y'). If the point be *on* the circle, the locus is (Art. 42) the tangent at the point (x', y').

NOTE.—The limits of this treatise forbid us from pursuing this subject further. The student who wishes to go on with it, is referred to more extended works on Conic Sections, such as Salmon's, Todhunter's, Puckle's, etc.

POLAR EQUATION.

50. *To find the polar equation of the circle.*

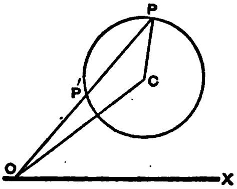

Fig. 49.

Let O be the pole, C the centre of the circle, and OX the initial line. Let the co-ordinates of the centre be the known quantities, r', θ', and the co-ordinates of any point P be r, θ, and R the radius of the circle. Then we have (Art. 14),

$$R = \sqrt{r^2 + r'^2 - 2rr' \cos(\theta - \theta')};$$

or

$$r^2 - 2rr' \cos(\theta - \theta') + r'^2 - R^2 = 0, \qquad (1)$$

which is the equation required.

Cor. 1.—Solving (1) for r, we obtain

$$r = r' \cos(\theta - \theta') \pm \sqrt{R^2 - r'^2 \sin^2(\theta - \theta')}. \qquad (2)$$

These two values of r in (2) are the two distances from the pole O to P and P', and are *real* and *unequal*, or *real* and *equal*, according as $r'^2 \sin^2(\theta - \theta') <$ or $= R^2$; or when $\sin^2(\theta - \theta') <$ or $= \frac{R^2}{r'^2}$. But when

$$\sin^2(\theta - \theta') = \frac{R^2}{r'^2},$$

we have

$$\sin(\theta - \theta') = \pm \frac{R}{r'},$$

showing that there are *two* positions in which r is tangent to the circle. The condition

$$\sin(\theta - \theta') = + \frac{R}{r'}$$

gives the upper point of tangency, for which $\theta > \theta'$. The condition

$$\sin(\theta - \theta') = - \frac{R}{r'}$$

gives the lower point of tangency, for which $\theta < \theta'$, or $\theta - \theta'$ is $-$, and hence $\sin(\theta - \theta')$ is $-$. From equation (2) we see that the two values of r have the same, or different signs, according as

$$\sqrt{R^2 - r'^2 \sin^2(\theta - \theta')} < \text{ or } > r' \cos(\theta - \theta').$$

In the former case, the pole is *without* the circle; in the latter it is *within*.

Cor. 2.—If $\theta' = 0$, the diameter is the initial line, and (1) becomes

$$r^2 - 2rr' \cos\theta + r'^2 - R^2 = 0. \qquad (3)$$

If, in addition, the pole be on the circumference, $r' = R$, and (3) becomes

$$r = 2R\cos\theta, \qquad (4)$$

a result which we might have obtained at once geometrically from the property that the inscribed angle in a semicircle is a right angle.

These polar equations may be deduced from the equations referred to rectangular axes (Art. 41) by putting $r\cos\theta$ and $r\sin\theta$ for x and y respectively. The student should deduce these equations by this method.

EXAMPLES.

1. Find the points where the axes are cut by

$$x^2 + y^2 - 5x - 7y + 6 = 0.$$

By making alternately $y = 0$, $x = 0$, in the given equation, we find that the points are determined by the quadratics

$$x^2 - 5x + 6 = 0, \qquad y^2 - 7y + 6 = 0,$$

giving us the points,

$$x = 3, \quad x = 2; \qquad y = 6, \quad y = 1.$$

2. Find the equation of the circle which touches the axes at the distance of 5 from the origin.

Ans. $x^2 + y^2 - 10x - 10y + 25 = 0$.

3. Find the equation of the circle whose centre is at the origin, and whose radius $= 3$, the axes being inclined at an angle of 45°.

Ans. $x^2 + y^2 + xy\sqrt{2} - 9 = 0$.

4. Find the equation of the circle whose centre is at $(-\frac{1}{3}, -\frac{1}{3})$, and whose radius $= \frac{2}{\sqrt{3}}$, the axes being inclined at an angle of 60°.

Ans. $x^2 + y^2 + xy + x + y - 1 = 0$.

5. Find the relation between a, b, r, in order that the line

$$\frac{x}{a} + \frac{y}{b} = 1 \tag{1}$$

may touch the circle $x^2 + y^2 = r^2$. (2)

Comparing (1) with (8) in Art. 42, we have

$$\frac{1}{a} = \frac{x'}{r^2} \quad \text{or} \quad \frac{x'}{r} = \frac{r}{a}; \qquad \text{and} \qquad \frac{1}{b} = \frac{y'}{r^2} \quad \text{or} \quad \frac{y'}{r} = \frac{r}{b};$$

Substituting these values of $\frac{x'}{r}$ and $\frac{y'}{r}$ in (4) of Art. 41, we have

$$\frac{r^2}{a^2} + \frac{r^2}{b^2} = 1; \qquad \text{or} \qquad \frac{1}{r^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

6. Find the equation of the circle whose centre is at the origin of co-ordinates, and which is touched by the line $y = 2x + 3$.

Ans. $x^2 + y^2 = \frac{9}{5}$.

7. On a circle whose radius $= 6$, a tangent is drawn at the point whose ordinate is 4. Find where the tangent cuts the two axes, and also determine the angle which it makes with the axis of x.

Ans. It cuts the axes at $\frac{8}{\sqrt{5}}$ and 9; angle $= \tan^{-1} \frac{1}{2}\sqrt{5}$.

8. Find the pole of $3x + 4y = 7$ with regard to

$$x^2 + y^2 = 14.$$

Ans. (6, 8), as appears from comparing the given equation with $xx' + yy' = r^2$.

9. Find the pole of $2x + 3y = 6$ with regard to

$$x^2 + y^2 = 18. \qquad \textit{Ans.}\ (6, 9).$$

10. Show that the polar of the point (x', y') with regard to the circle $(x - a)^2 + (y - b)^2 = r^2$ is

$$(x - a)(x' - a) + (y - b)(y' - b) = r^2.$$

11. Find the polar of (4, 4) with regard to

$$(x - 1)^2 + (y - 2)^2 = 13.$$

Ans. $3x + 2y = 20$.

12. Find the polar of (4, 5) with regard to

$$x^2 + y^2 - 3x - 4y = 8.$$

Ans. $5x + 6y = 48$.

13. Find the pole of $2x + 3y = 6$ with regard to

$$(x - 1)^2 + (y - 2)^2 = 12.$$

Ans. $(-11, -16)$.

14. Find the polar equation of the circle whose centre is at $\left(8, \frac{\pi}{4}\right)$, and whose radius is 10; and determine where the circle cuts the initial line.

Ans. Equation is $r^2 - 8\sqrt{2}(\sin\theta + \cos\theta)r = 36$; cuts the initial line at $r = (4 \pm \sqrt{34})\sqrt{2}$.

15. Find the polar equation of the circle whose centre is $\left(15, \frac{\pi}{2}\right)$, and whose radius is 10; and determine the values of θ when the radius-vector is tangent to the circle.

$$\textit{Ans.} \begin{cases} \text{Equation is } r^2 - 30r\sin\theta = -125; \\ \theta = \cos^{-1}(\pm \frac{2}{3}). \end{cases}$$

16. Determine what is represented by the equation

$$r^2 - ra \cos 2\theta \sec \theta - 2a^2 = 0.$$

Ans. $\begin{cases} \text{A circle whose equation is} & r = 2a \cos \theta, \\ \text{and a right line whose equation is} & r = -a \sec \theta. \end{cases}$

17. Determine the radius and the centre of the circle

$$r^2 - 2r(\cos \theta + \sqrt{3} \sin \theta) = 5.$$

[Compare with (1) in Art. 50.]

Ans. Radius $= 3$; $r' = 2$, $\theta' = \frac{\pi}{3}$.

18. A limited right line moves so that its extremities are always on the co-ordinate axes; show that the locus of its middle point is a circle.

19. Show what the equation of the circle becomes when the origin is on the circumference, and the axes are inclined at an angle of 120°, the parts of them intercepted by the circle being h and k.

Since the origin is on the curve, the absolute term is zero (Art. 42, Cor. 2); therefore the equation of the circle referred to oblique axes (Art. 42), when expanded, becomes

$$x^2 + y^2 + 2xy \cos \omega - 2(a + b \cos \omega)x - 2(b + a \cos \omega)y = 0. \quad (1)$$

Making alternately $y = 0$, $x = 0$, we have, for determining the intercepts on the two axes,

$$x^2 - 2(a + b \cos \omega)x = 0,$$
$$y^2 - 2(b + a \cos \omega)y = 0.$$

$$\therefore \quad x = 2(a + b \cos \omega) = h,$$
$$y = 2(b + a \cos \omega) = k.$$

When $\omega = 120°$, $\cos \omega = -\frac{1}{2}$; $\therefore$ (1) becomes,

$$x^2 + y^2 - xy - hx - ky = 0, \quad \textit{Ans.}$$

20. Find the inclination of the axes in order that the equation

$$x^2 + y^2 - xy - hx - hy = 0$$

may represent a circle; and find the length of the radius and the co-ordinates of the centre.

[Compare with (2), Art. 42.]

$$\textit{Ans.} \begin{cases} \omega = 120^\circ; \\ \text{co-ordinates of the centre} = h; \\ \text{radius} = h. \end{cases}$$

21. Find the inclination of the axes in order that the equation

$$x^2 + y^2 + xy - hx - hy = 0$$

may represent a circle, and find the length of the radius and the position of the centre.

Ans. $\omega = 60^\circ$; $a = b = \frac{h}{3}$; $r = \frac{h}{\sqrt{3}}$.

22. Given the base of a triangle $= 2m$, and the sum of the squares on its sides $= 2s^2$, to find the locus of its vertex.

[Take the base and a perpendicular through its centre for axes.] *Ans.* $x^2 + y^2 = s^2 - m^2$.

23. A point moves so that the sum of the squares of its distances from the four sides of a square is constant; show that the locus of the point is a circle.

CHAPTER V.

THE PARABOLA.

51. In the previous chapters we investigated various properties of right lines and circles. We shall now proceed to consider three curves, commonly called *conic sections*, which rank next in importance and interest to the right line and circle.

A **Conic Section** is the locus of a point moving in a plane so that its distance from a fixed point bears a constant ratio to its distance from a fixed right line. If this ratio is *unity*, the locus is a **Parabola**; if *less* than unity an **Ellipse**; if *greater* than unity, an **Hyperbola**. The fixed point is called the **Focus**, and the fixed right line is called the **Directrix**.*

We might begin by producing the general equation of a conic section, and afterwards applying it to the parabola, ellipse, and hyperbola, in succession; † but we prefer to find the equation of each conic section separately from its definition, beginning with the *parabola*, because it is the simplest of the three.

REMARK.—It will be shown hereafter, that if a right cone with a circular base be cut by a plane, the curve of intersection will be one of the following: a parabola, an ellipse, an hyperbola, a circle, one right line, two right lines, or a point. Hence, the parabola, ellipse, and hyperbola are called conic sections, which term may also be extended to include the circle, one right line, two right lines, and the point. It was from this point of view that these curves were first examined by geometers. It will be shown hereafter that every equation of the second degree between two variables is the equation of a conic section.

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* Todhunter's Conic Sections, p. 116.

† See O'Brien's Co-ordinate Geometry, p. 62.

52. A **Parabola** is the locus of a point moving in a plane so that its distance from a fixed point is equal to its distance from a fixed right line. The fixed point is called the **Focus**; the fixed right line is called the **Directrix**; the right line through the focus perpendicular to the directrix is called the **Axis** of the curve; the point in which the axis cuts the curve is called the **Principal Vertex.**

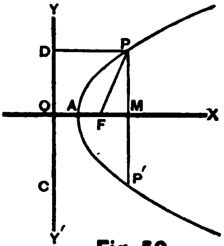

Fig. 50.

From the definition, the parabola may readily be constructed by points, thus: Let F be the focus, CD the directrix, and OX through F perpendicular to CD the axis. The point A, midway between O and F, is a point of the curve, and is the *vertex*. Take any point on the axis, as M, and erect MP perpendicular to it. With F as a centre and OM as a radius, describe an arc cutting MP at P. This will be a point of the curve, for we have FP = DP. In the same way, any number of points may be constructed; drawing a line through them, it will be the required curve.

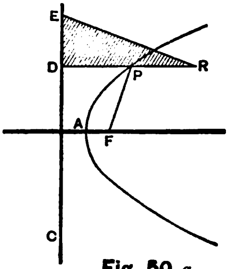

Fig. 50.*a*

The curve may also be described by a continuous movement. Let CD be the directrix and F the focus. Take a triangular ruler, RDE, right-angled at D, and place one side DE on the directrix; take a string, equal in length to RD, and attach one end at R, and the other at F; then press a pencil against the string, keeping it continually tight, with the point P against the ruler, and slide the ruler along the directrix; the path of the pencil will be a parabola, for in every position of P we shall have

$$PD = FP.$$

53. *To find the equation of the parabola.*

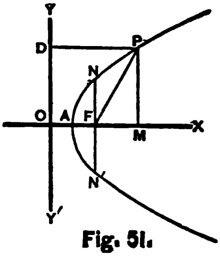

Fig. 51.

Let F be the focus, YY′ the directrix, OX the axis of the parabola. Take OX and OY for the co-ordinate axes. Let x, y be the co-ordinates of any point P in the locus, and put $p =$ the constant distance OF. Draw PM and PD respectively perpendicular to the axes of x and y, and join FP. Then we have, from the definition,

$$\text{FP} = \text{PD};$$

therefore, $\text{FM}^2 + \text{MP}^2 = \text{PD}^2;$

that is, $(x - p)^2 + y^2 = x^2,$

or $$y^2 = 2p\,(x - \tfrac{1}{2}p), \qquad (1)$$

which is true for every position of P; hence it is the equation required.

When $y = 0$, $x = \frac{1}{2}p$, which shows that the curve cuts the axis of x at the distance $\frac{1}{2}p$ to the right of the origin, or midway between O and F.

If we move the origin to A, and keep the new axes parallel to the old, the equation will be simplified. The formulæ for transformation (Art. 33) are $x = \frac{1}{2}p + x'$, $y = y'$; therefore (1) becomes

$$y'^2 = 2px';$$

or removing the accents, since x and y are general variables, we have

$$y^2 = 2px, \qquad (2)$$

which is the equation of the parabola referred to its axis and the tangent at the principal vertex.

Cor. 1.—When $y = 0$ in (2), we have $x = 0$, which shows that the curve cuts the axis of x at the origin. When $x = 0$, $y = \pm 0$, which shows that the axis of y is tangent to the curve at the origin.

COR. 2.—Solving (2) for y, we get

$$y = \pm \sqrt{2px},$$

which shows that for positive values of x there are ***two real*** values of y, numerically equal, but with contrary signs. Hence, for every point P on one side of the axis of x, there is a point P′ on the other side, at the same distance from it; and therefore the curve is symmetrical with respect to the axis of x. If we suppose p to be positive, which is the case when the focus is to the *right* of the origin, we see that negative values of x do not give *real* values of y; hence, no point of the curve lies to the left of the axis of y. As x may have any positive value whatever, the curve extends to an infinite distance in the direction of positive abscissas. In the same way, if we suppose p to be negative, or the focus to be to the *left* of the origin, it may be shown that no part of the curve lies to the right of the origin, while it extends without limit to the left of it.

COR. 3.—To find the value of the ordinate passing through the focus, make $x = \frac{1}{2}p$, and get, from (2),

$$y^2 = p^2, \qquad \text{or} \qquad 2y = 2p.$$

Hence, the double ordinate passing through the focus is equal to the constant quantity $2p$. This double ordinate through the focus of a conic section is called the **Principal Parameter**, or **Latus Rectum**.

COR. 4.—From (2) we have the proportion,

$$x : y :: y : 2p;$$

that is, $2p$, *the latus rectum, is a third proportional to any abscissa and its corresponding ordinate.*

COR. 5.—If (x', y') and (x'', y'') be any two points on the curve, we shall have, from equation (2),

$$y'^2 = 2px'; \qquad y''^2 = 2px''.$$

Hence, forming a proportion, we have

$$y'^2 : y''^2 :: x' : x''.$$

That is, *the squares ef any two ordinates are to each other as their corresponding abscissas.*

Cor. 6.—A point is *outside, on,* or *inside* the parabola, according as

$$y^2 - 2px >, =, \text{ or } < 0.$$

Thus, if the point is *on* the curve, as at P, its co-ordinates satisfy the equation of the curve, giving

$$y^2 - 2px = 0.$$

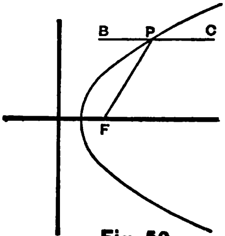

Fig. 52.

If the point is *outside* of the curve, as at B, its abscissa will be less than at P, while its ordinate will be the same, giving

$$y^2 - 2px > 0.$$

If the point is *inside* of the curve, as at C, its abscissa will be greater than at P, while its ordinate will be the same, giving

$$y^2 - 2px < 0.$$

54. *To find the equation of the tangent at any point of a parabola* (see Def., Art. 42).

Let (x', y') and (x'', y'') be any two points on the curve. The equation of the secant through these points is (Art. 26)

$$y - y' = \frac{y' - y''}{x' - x''}(x - x'). \qquad (1)$$

Since (x', y') and (x'', y'') are on the parabola, they will satisfy its equation, giving us

$$y'^2 = 2px', \qquad (2)$$

$$y''^2 = 2px''. \qquad (3)$$

Subtracting (3) from (2), factoring, and dividing, we have

$$\frac{y'-y''}{x'-x''}=\frac{2p}{y'+y''},$$

which, substituted in (1), gives

$$y-y'=\frac{2p}{y'+y''}(x-x'). \qquad (4)$$

When the points become consecutive, $y''=y'$; hence (4) becomes

$$y-y'=\frac{p}{y'}(x-x'). \qquad (5)$$

Clearing of fractions, and substituting for y'^2 its value in (2), we have

$$yy'=p(x+x'), \qquad (6)$$

which is the required equation of the tangent at (x', y').

This equation may be written

$$y=\frac{p}{y'}(x+x'), \qquad (7)$$

in which $\frac{p}{y'}$ is the tangent of the angle which the tangent line at the point (x', y') makes with the axis of x.

COR.—To find the point in which the tangent cuts the axis of x, make $y=0$, in (6), and we have

$$0=p(x+x'); \qquad \therefore \quad x=-x';$$

that is, *the subtangent is bisected at the vertex.*

SCH.—This result enables us to draw a tangent to the curve at a given point. Let P be the given point, and MP its ordinate. Lay off AT to the left of the origin equal to AM. Draw a line through T and P, and it will be the tangent required.

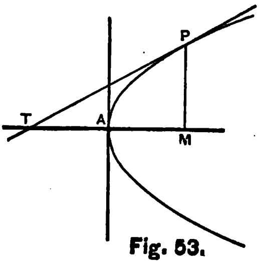

Fig. 53.

55. *To find the equation of the normal at any point of a parabola.*

Let (x', y') be the point; the equation of the tangent at (x', y'), (Art. 54), is

$$y = \frac{p}{y'}(x + x'). \tag{1}$$

The equation of a right line through (x', y') perpendicular to (1) is, by Art. 27, Cor. 2,

$$y - y' = -\frac{y'}{p}(x - x'), \tag{2}$$

which is the required equation of the normal at the point (x', y').

Cor.—To find the point in which the normal at (x', y') cuts the axis of x, we make $y = 0$ in (2), and get, after reduction,

$$x = x' + p; \qquad \text{or} \qquad x - x' = p.$$

That is, *the subnormal is constant, and equal to half the latus rectum.*

Sch. — This furnishes a second method of drawing a tangent to a parabola, at a given point.

Let P be the given point, and PM its ordinate. From the foot of the ordinate lay off a distance MG on the axis, to the right, equal to half the latus rectum, and draw GP; through P draw PT perpendicular to GP. PT will be the tangent required, and GP will be the normal.

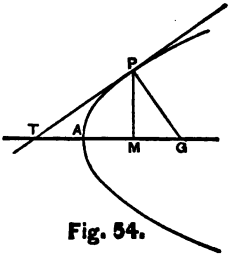

Fig. 54.

56. *To prove that a tangent to the parabola at any point makes equal angles with the axis of the curve and the focal line to the point of contact.*

A **Focal Line** is a line drawn from the focus to a point of the curve.

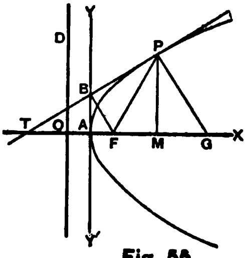

Fig. 55.

Let PT be the tangent at P, FP the focal line to the point of contact, MP the ordinate, GP the normal, and OD the directrix. Then (Art. 54, Cor.),

$$AT = AM;$$

also,

$$\begin{aligned} FT &= AT + AF \\ &= AM + AF \\ &= AM + AO = OM; \end{aligned}$$

that is,

$$FT = FP \text{ (Art. 52).}$$

Hence the angle FTP = angle FPT.

57. *To find the locus of the intersection of the tangent at any point with the perpendicular on it from the focus.*

Let (x', y') be any point on the curve; the equation of the tangent at (x', y') is (Art. 54),

$$y = \frac{p}{y'}(x + x'). \tag{1}$$

The equation of the line through the focus perpendicular to (1) is (Art. 27),

$$y = -\frac{y'}{p}(x - \tfrac{1}{2}p). \tag{2}$$

We must now eliminate x' and y' by means of (1) and (2), and

$$y'^2 = 2px'. \tag{3}$$

Substituting in (1) the value of x' obtained from (3), we have

$$y = \frac{p}{y'}x + \frac{y'}{2}. \tag{4}$$

From (2) we have $y' = -\dfrac{py}{x - \frac{1}{2}p}$; (5)

which in (4) gives

$$y = -\frac{(x - \frac{1}{2}p)\,x}{y} - \frac{py}{2x - p};$$

which reduces to

$$[y^2 + (x - \tfrac{1}{2}p)^2]\,x = 0. \qquad (6)$$

This equation represents two loci,

$$x = 0 \qquad \text{and} \qquad y^2 + (x - \tfrac{1}{2}p)^2 = 0.$$

The first is the axis of y; and the second is the point $x = \frac{1}{2}p$, $y = 0$, since only the point $(\frac{1}{2}p, 0)$ will satisfy the equation $y^2 + (x - \frac{1}{2}p)^2 = 0$.

The co-ordinates of the former equation, $x = 0$, will satisfy (1) and (2), giving, in each equation, $y = \frac{1}{2}y$; it is therefore the locus of the intersection of (1) and (2). Hence the required locus is the tangent to the curve at the origin or vertex.

The latter equation is the focus, and is *not* the locus of the intersection of (1) and (2), since its co-ordinates will not satisfy (1), although they will satisfy (2).

Sch.—We have arrived here at two results, one which we are seeking, and another which does not satisfy the geometric conditions of the problem. It is frequently the case in algebraic questions that we arrive at more results than we are especially seeking. The values of x and y which satisfy equations (1) and (2) will satisfy (6); and therefore (6) represents a locus which passes through all the points where (1) and (2) intersect (Art. 30). It does *not* follow that every point whose co-ordinates satisfy (6) should be one of these points of intersection.

Cor.—The result of this Art. can be easily obtained from geometric considerations. Thus, let FB, Fig. 55, be a perpendicular from the focus to the tangent PT. It will intersect PT at its middle point B, because the triangle TFP is isosceles. The vertical tangent at A also intersects TP at its middle point B, because it bisects MT and is parallel to MP. Therefore the point B, at which the perpendicular intersects the tangent, is on the axis of y, or the tangent to the curve at the vertex.

58. *To find the co-ordinates of the point of contact of a tangent to a parabola from a fixed point.*

Let (x', y') be the required point of contact, and (x'', y'') be the fixed point through which the tangent passes.

Since (x', y') is on the parabola, we have

$$y'^2 = 2px'. \tag{1}$$

The equation of the tangent at (x', y') is

$$yy' = p(x + x').$$

Since this tangent passes through (x'', y''), we have

$$y''y' = p(x'' + x'). \tag{2}$$

Solving (1) and (2) for x' and y', we have

$$x' = y''^2 - 2px'' \pm y''\sqrt{y''^2 - 2px''},$$
$$y' = y'' \pm \sqrt{y''^2 - 2px''}.$$

These values indicate that from any fixed point *two* tangents can be drawn to a parabola, *real*, *coincident*, or *imaginary*, according as $y''^2 - 2px'' > 0$, $= 0$, or < 0; that is, according as the point (x'', y'') is *without*, *on*, or *within* the curve (see Art. 53, Cor. 6).

Cor.—The ordinate of the middle point of the chord joining the *two real* points of contact is equal to the half-sum of the ordinates of the two points; that is, it is equal to y''. Hence, a line through the fixed point, parallel to the axis of the curve, bisects the chord joining the two points of contact. This chord is called the **Chord of Contact.**

59. *Tangents are drawn to a parabola from a given external point; to find the equation of the chord of contact.*

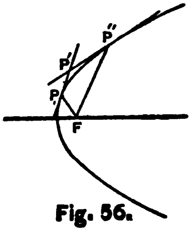

Fig. 56.

Let (x', y') be the external point P'; (x'', y'') and (x_1, y_1) the two points P'' and P_1 where the tangents meet the parabola. Then $P''P_1$ will be the chord of contact whose equation is required.

The equation of the tangent at P'' (Art. 54) is

$$yy'' = p(x + x''). \tag{1}$$

Since this tangent passes through P', we have

$$y'y'' = p(x' + x''). \tag{2}$$

Also the equation of the tangent at P_1 is

$$yy_1 = p(x + x_1). \tag{3}$$

Since this tangent passes through P', we have

$$y'y_1 = p(x' + x_1). \tag{4}$$

Hence it follows that the equation of the chord of contact $P''P_1$ is

$$yy' = p(x + x'). \tag{5}$$

For (5) is the equation of *some right line* (Art. 23); and this line passes through P'', for (5) is satisfied by the co-ordinates of P'', $x = x''$, $y = y''$, as we see from (2). Also, the line passes through P_1, for (5) is satisfied by the co-ordinates of P_1, $x = x_1$, $y = y_1$, as we see from (4). Hence, (5) is the required equation.

60. *Through any fixed point a chord is drawn to a parabola, and tangents to the parabola are drawn at the extremities of the chord; to find the equation of the locus of the intersection of the tangents when the chord is turned about the fixed point.*

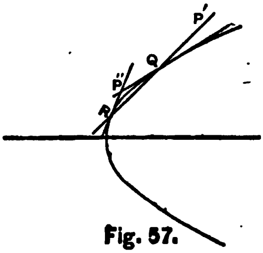

Fig. 57.

Let (x', y') be the fixed point P' through which the chord passes; and (x'', y'') the point P'' in which the two tangents drawn at the extremities Q and R of the chord, intersect.

The equation of the chord of contact (Art. 59) is

$$yy'' = p(x + x''). \tag{1}$$

Since this chord passes through P', we have

$$y'y'' = p(x' + x''). \tag{2}$$

Now, if the chord turns about the fixed point (x', y'), the intersection (x'', y'') of the corresponding tangents is *any* point in the required locus; that is, the co-ordinates x'', y'' of any point in the required locus satisfy the equation

$$yy' = p(x + x'), \tag{3}$$

which is therefore the equation required, and the locus is a right line.

Sch.—The line $yy' = p(x + x')$ is called the **Polar** of the point (x', y') with regard to the parabola $y^2 = 2px$, and the point (x', y') is called the **Pole** of the line.

It will be seen (Art. 60), that if (x', y') be *any point whatever*, the equation $yy' = p(x + x')$ represents the locus of the intersection of the tangents at the extremities of the chord through (x', y').

61. The statements in Art. 49 with respect to the circle may all be applied to the parabola. Thus, we see that the equations of the *tangent*, of the *chord of contact*, and of the *locus of the intersection of tangents at the extremities of chords that pass through a fixed point*, are all identical in form; and inasmuch as the fixed point (x', y'), in the case of the chord of contact, is restricted to being *without* the curve, and in that of the tangent to being *on* the curve, while in the case of the locus just described it is *any point whatever,* it follows that the tangent and chord of contact in the parabola are particular cases of the locus, due to bringing the point (x', y') *on* the curve, or *outside* of it.

62. *A* **Diameter** *of a curve is the locus of the middle points of parallel chords:*

To find the equation of any diameter.

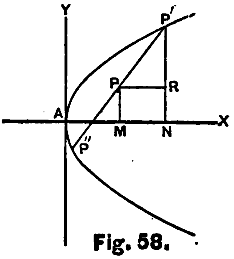

Fig. 58.

Let (x, y) be the middle point P of the chord P′P″; (x', y') the point P′ or P″; θ the inclination of P′P″ to the axis of x, the axis of the curve; r the length of PP′, half the line P′P″. Then

$$x' = AM + MN = x + r\cos\theta;$$

$$y' = NR + RP' = y + r\sin\theta.$$

Now as P′ is on the curve, its co-ordinates x', y' will satisfy the equation of the curve $y^2 = 2px$, giving

$$(y + r\sin\theta)^2 = 2p\,(x + r\cos\theta),$$

or $$r^2\sin^2\theta + 2r\,(y\sin\theta - p\cos\theta) + y^2 - 2px = 0, \quad (1)$$

from which quadratic we can determine the two values of r. But as (x, y) is the middle point P of the chord, the two values of r are numerically equal with contrary signs; therefore (from Algebra), the coefficient of the first power of r vanishes, giving us

$$y\sin\theta - p\cos\theta = 0,$$

which represents the locus of the middle point P of the chord P′P″. Hence the required equation of any diameter is

$$y' = p\cot\theta. \quad (2)$$

Since p is fixed for any given parabola, and θ is constant for any given system of parallel chords, the second member of (2) is constant; and therefore it is a right line parallel to the axis of x (Art. 22, I, Cor. 2). Hence, every diameter is a right line parallel to the axis of the parabola. By giving to θ a suitable value, equation (2) may be made to represent *any* right line parallel to the axis. Hence it follows that every right line parallel to the axis of the parabola is a diameter; that is, it bisects some system of parallel chords.

SCH.—To draw a diameter of a parabola, draw any two parallel chords, bisect them; the line passing through the points of bisection is a diameter.

63. *To find the equation of the parabola referred to any diameter and the tangent at its vertex.*

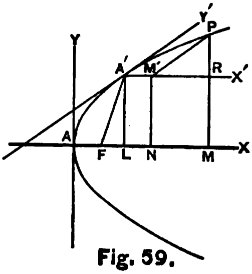

Fig. 59.

Let (m, n) be any point A′ on the parabola; take this point for the new origin, and draw through it the diameter A′X′ and the tangent A′Y′ for the new axes of co-ordinates. Let $X'A'Y' = \beta$; then (Art. 55),

$$\tan\beta = \frac{p}{y'} = \frac{p}{n}.$$

Let (x, y) be any point P on the curve referred to the old axes AX and AY. Draw PM′ parallel to A′Y′, and draw PM, M′N, and A′L parallel to AY; then

$$x = AM = AL + A'M' + M'R = m + x' + y'\cos\beta. \qquad (1)$$

$$y = MP = LA' + RP \qquad = n + y'\sin\beta. \qquad (2)$$

Substitute these values of x and y in the equation

$$y^2 = 2px,$$

and obtain $(n + y'\sin\beta)^2 = 2p\,(m + x' + y'\cos\beta)$, or $y'^2 \sin^2\beta + 2y'(n\sin\beta - p\cos\beta) + (n^2 - 2pm) = 2px'$. (3)

But
$$n = \frac{p}{\tan\beta} = p\frac{\cos\beta}{\sin\beta};$$
$$\therefore \quad n\sin\beta - p\cos\beta = 0;$$
also, since A′ is on the curve, its co-ordinates m, n will satisfy $y^2 = 2px$, giving us,
$$n^2 = 2pm.$$

Hence (3) becomes $y'^2 \sin^2\beta = 2px'$,

or
$$y'^2 = \frac{2p}{\sin^2\beta}x'.$$

Putting $\frac{2p}{\sin^2\beta} = 2p'$, and dropping the accents from x and y, since they are general variables, we have
$$y^2 = 2p'x, \tag{4}$$
which is the required equation, and is of the same form as the corresponding equation referred to the axis of the curve and the tangent at the principal vertex.

Sch.—We might have obtained equations (1) and (2) from the formulæ to pass from rectangular axes to oblique (Art. 35, Cor. 1), by remembering that, since the new axis of x is parallel to the old, $\alpha = 0$, and therefore $\sin\alpha = 0$, and $\cos\alpha = 1$.

Cor. 1.—Solving equation (4) for y, we have
$$y = \pm\sqrt{2p'x},$$
which shows that, for every positive value of x, there are two *real* values for y, numerically equal, but with contrary signs. These two values, taken together, make up a chord parallel to the axis of y, and which is bisected by the axis of x. Hence the axis of x bisects all chords of the curve parallel to the axis of y; that is, the system of chords bisected by any diameter (Art. 62), is parallel to the tangent at the vertex.

The quantity $2p'$, or its equal $\dfrac{2p}{\sin^2\beta}$, is called the **Parameter** of the diameter that is taken as the axis of x.

Cor. 2.—If (x', y') and (x'', y'') be any two points on the curve, we have from (4),

$$y'^2 = 2p'x'; \qquad y''^2 = 2p'x'';$$

therefore, $$y'^2 : y''^2 :: x' : x''.$$

That is, the squares of the ordinates to any diameter are to each other as the corresponding abscissas.

64. *The parameter of any diameter is equal to four times the distance from the vertex of that diameter to the focus.*

By Art. 56 we have, in Fig. 59,

$$\mathrm{FA'} = \mathrm{AL} + \mathrm{AF} = m + \tfrac{1}{2}p;$$

and by Art. 63,

$$m = \frac{n^2}{2p} = \tfrac{1}{2}p\cot^2\beta \text{ (since } n = p\cot\beta);$$

therefore $$m + \tfrac{1}{2}p = \tfrac{1}{2}p\cot^2\beta + \tfrac{1}{2}p = \frac{p}{2\sin^2\beta} = \mathrm{FA'};$$

hence, $$\frac{2p}{\sin^2\beta} = 4\mathrm{FA'}.$$

But (Art. 63, Cor. 1), $\dfrac{2p}{\sin^2\beta}$ is the parameter of the diameter A′X′, which was represented by $2p'$ · therefore the parameter of any diameter is equal to 4FA′.

65. *To find the equation of a tangent to a parabola referred to any diameter and the tangent at its vertex.*

The equation of a right line referred to oblique axes is of the same form (Art. 22, IV) as when referred to rectangular axes; also the equation of the parabola referred to any diameter and the tangent at its vertex is of the same form (Art. 63) as when referred to the axis of the curve and tangent at the principal vertex. Hence, the investigation of Art. 54 will apply without any change to the equation

$$y^2 = 2p'x,$$

giving us the required equation,

$$yy' = 2p'(x + x').$$

COR.—Making $y = 0$ in this equation, we get $x = -x'$, which shows that the tangent cuts any diameter on the left of its vertex, at a distance equal to the abscissa of the point of contact. Hence, *the subtangent to any diameter of a parabola is bisected at the vertex.*

66. *To find the polar equation of the parabola, the focus being the pole.*

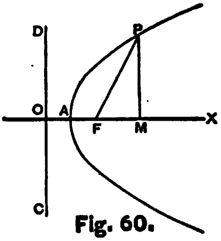

Fig. 60.

Let $FP = r$, $XFP = \theta$; then we have, from the definition of Art. 52,

$$FP = OM = OF + FM;$$

that is, $$r = p + r\cos\theta;$$

therefore, $$r = \frac{p}{1 - \cos\theta}, \qquad (1)$$

which is the required equation.

COR.—When $\theta = 0$, $r = \frac{p}{1-1} = \infty$, which shows that the radius-vector which coincides with the axis does not meet the curve, or rather meets it at an infinite distance. For any value of $\theta > 0$, however small, r is finite, which shows that if a line be drawn from the focus making any angle, however small, with the axis of the curve, it will meet the curve at a finite distance.

When $\theta = 90°$, $r = p$, as it should. When $\theta = 180°$, $r = \frac{1}{2}p$, as it should, since $\text{AF} = \frac{1}{2}p$ (Art. 53). When $\theta = 270°$, $r = p$, as it should. The two values of r corresponding to 90° and 270°, taken together, make the *parameter* of the axis of the curve, which is again seen to be equal to $2p$, as it was shown to be in Art. 53, Cor. 3.

67. *A chord passing through the focus of a conic section is called a* **Focal Chord.**

If tangents are drawn at the extremities of any focal chord of a parabola:

I. *The tangents will intersect on the directrix.*

II. *The tangents will meet at right angles.*

III. *The line drawn from the point of intersection of the tangents to the focus will be perpendicular to the focal chord.*

I. If the tangents to a parabola meet at the point (x', y'), the equation of the chord of contact (Art. 59) is

$$yy' = p\,(x + x').$$

If the chord passes through the focus, its co-ordinates, $x = \frac{1}{2}p$, $y = 0$, must satisfy this equation, giving

$$0 = p\,(\tfrac{1}{2}p + x'); \qquad \therefore\ x' = -\tfrac{1}{2}p;$$

that is, the point of intersection of the tangents is on the directrix.

II. The equation of the tangent to a parabola (Art. 54) is

$$y = \frac{p}{y'}(x + x') = \frac{p}{y'}x + \frac{p}{y'}x' = \frac{p}{y'}x + \frac{\frac{1}{2}y'^2}{y'} = \frac{p}{y'}x + \frac{y'}{2}. \quad (1)$$

$$\text{Put} \qquad \frac{p}{y'} = a, \qquad \text{and} \qquad \therefore\ \frac{y'}{2} = \frac{p}{2a};$$

now substitute these values in (1), and we have

$$y = ax + \frac{p}{2a}. \qquad (2)$$

If the tangents meet at (x', y'), we have

$$y' = ax' + \frac{p}{2a};$$

or
$$a^2 - \frac{y'}{x'}a + \frac{p}{2x'} = 0, \tag{3}$$

a quadratic for determining the two values of a, which are the tangents of the angles that the two tangent lines through (x', y') make with the axis of the parabola.

Call the two roots of (3), a_1 and a_2, and we have from Algebra,

$$a_1 a_2 = \frac{p}{2x'}. \tag{4}$$

From (I) we have $x' = -\frac{1}{2}p$, which in (4) gives

$$a_1 a_2 = -1, \qquad \text{or} \qquad a_1 = -\frac{1}{a_2};$$

that is, the two tangents are perpendicular to each other (Art. 27, Cor. 1).

III. The equation of the right line passing through the focus and the point (x', y'), by Art. 26, is

$$y = \frac{y'}{x' - \frac{1}{2}p}(x - x'). \tag{5}$$

From (I), $x' = -\frac{1}{2}p$, which in (5) gives

$$y = -\frac{y'}{p}(x - x'). \tag{6}$$

The equation of the chord of contact (Art. 59) is

$$yy' = p(x + x'),$$

which becomes for the focal chord,

$$y = \frac{p}{y'}(x - \tfrac{1}{2}p), \tag{7}$$

which is perpendicular to (6), by Art. 27, Cor. 1.

EXAMPLES.

1. Find the intersections of the parabola $y^2 = 8x$ and the line $3y - 2x - 8 = 0$. *Ans.* $(2, 4)$ and $(8, 8)$.

2. Find the equation of the right line passing through the focus of the parabola $y^2 = 4x$, and making an angle of 45° with the axis of the curve. *Ans.* $y = x - 1$.

3. Find the points in which the focal chord, $y = x - 1$, intersects the parabola, $y^2 = 4x$.

Ans. $(3 \pm 2\sqrt{2},\ 2 \pm \sqrt{2})$.

4. Find the equation of the right line passing through the vertex of any parabola and the extremity of the focal ordinate. *Ans.* $y = 2x$.

5. Find the equation of the circle which passes through the vertex of any parabola and the extremities of the double ordinate through the focus. *Ans.* $y^2 = \frac{5}{2}px - x^2$.

6. Find the equation of the circle which passes through the vertex of the parabola $y^2 = 12x$ and the extremities of the double ordinate through the focus.

Ans. $y^2 = 15x - x^2$.

7. Find the equations of the tangent and normal to any parabola at the extremity of the positive ordinate through the focus. *Ans.* $y = x + \frac{1}{2}p$ and $y + x = \frac{3}{2}p$.

8. Find the equations of the tangent and normal to the parabola $y^2 = 4x$, at the extremity of the positive ordinate through the focus. *Ans.* $y = x + 1$; $y + x = 3$.

9. Find the point where the normal in Ex. 7 meets the curve again, and the length of the intercepted chord.

Ans. $(\frac{9}{2}p,\ -3p)$; length of chord $= 4p\sqrt{2}$.

10. Find the point where the normal in Ex. 8 meets the curve again, and the length of the intercepted chord.

Ans. $(9,\ -6)$; length of chord $= 8\sqrt{2}$.

11. Find the point in a parabola where the tangent is inclined at an angle of 30° to the axis of x.
Ans. $(\frac{3}{2}p,\ p\sqrt{3})$.

12. Prove that the normal at any point of a parabola bisects the angle between the focal line and the diameter passing through that point. [See Art. 56.]

13. Prove that the quantity $2p'$, in equation (4) of Art. 63, which in the corollary of that article was called the *parameter*, is equal to the double ordinate passing through the focus. [See Art. 64.]

14. On a parabola whose latus rectum is 10, a tangent is drawn at the point whose ordinate is 6, the origin being at the principal vertex; determine where the tangent cuts the two co-ordinate axes. *Ans.* $(-3.6,\ 0)$ and $(0,\ 3)$.

15. Determine where the normal in the preceding example, at the same point, if produced, will cut the two axes.
Ans. $(8.6,\ 0)$ and $(0,\ 10.3)$.

16. Find the angle which the tangent in Ex. 14 makes with the axis of x. *Ans.* 39° 48′ 20″.

17. In the parabola $y^2 = 12x$, find the length of the perpendicular from the focus to the tangent at the point whose abscissa is 9. *Ans.* 6.

18. In the parabola $y^2 = 8x$, find the length of the normal at the point whose abscissa is 6. *Ans.* 8.

19. Prove that the circle described on a focal chord as a diameter is tangent to the directrix.

20. Prove that the tangent at any point of a parabola will meet the directrix and latus rectum produced, at two points equally distant from the focus.

21. Prove that a right line drawn from the point of the parabola of which the abscissa is $4p$, and cutting the axis at the point $x = 2p$, will, if produced, meet the curve again at

the point $x = p$, and be a normal at that point, $2p$ being the latus rectum.

22. The extremities of any chord of a parabola being (x', y'), (x'', y''), and the abscissa of its intersection with the axis of the curve being x, to prove that

$$x'x'' = x^2, \qquad y'y'' = -4px.$$

23. Two tangents of a parabola meet the curve in (x', y') and (x'', y''), their point of intersection being (x, y); show that

$$x = \sqrt{x'x''}, \qquad y = \frac{y' + y''}{2}.$$

24. The latus rectum of a parabola is 10, and the radius-vector is 25; find the variable angle. *Ans.* 143° 7′ 48″.

25. The latus rectum of a parabola is 10, and the variable angle is 144°, the pole being at the focus; determine the radius-vector. *Ans.* 26.18.

CHAPTER VI.

THE ELLIPSE.

68. The **Ellipse** is the locus of a point moving in a plane, so that its distance from a fixed point bears a constant ratio to its distance from a fixed right line, the ratio being less than unity.*

From this definition the ellipse may be constructed by points, thus:

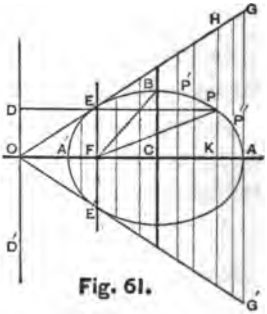

Fig. 61.

Let F be the fixed point, DD′ the fixed right line, and e the given ratio. Draw through F the line OA perpendicular and EE′ parallel to DD′. Take

$$FE\ (= FE') : FO \ :: \ e : 1,$$

and draw OE and OE′ produced indefinitely. Draw parallels to EE′, meeting the lines OG and OG′. With the half of any one of these lines, as KH, for a radius, and the fixed point F for a centre, describe an arc cutting KH at P; this is a point of the curve; for, joining P and F, and drawing PD perpendicular to DD′, we have $KH\ (= FP) : KO\ (= PD) \ :: \ FE : FO$.

That is, by construction we have, $FP : PD \ :: \ e : 1$.

In the same way, any required number of points in the curve may be found. Connecting these points by a line, we have the required ellipse. The fixed line DD′ is called the **Directrix**; the fixed point F is called the **Focus**; OG and OG′ are the **Focal Tangents**.

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* Todhunter's Conic Sections, p. 143.

69. *To find the equation of the ellipse.*

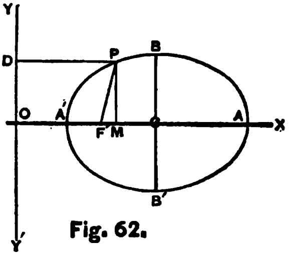

Fig. 62.

Let F' be the focus; YY' the directrix. Through F' draw OX perpendicular to YY'; take OX and OY for the co-ordinate axes.

Let (x, y) be any point P on the locus; join F'P; draw PM and PD respectively perpendicular to OX and OY.

Represent OF' by p, and the ratio of F'P to PD by e. Then we have, from definition,

$$F'P = e \cdot PD,$$

or
$$\overline{F'P}^2 = e^2 \cdot \overline{PD}^2,$$

or
$$\overline{F'M}^2 + \overline{MP}^2 = e^2 \cdot \overline{PD}^2;$$

that is,
$$(x - p)^2 + y^2 = e^2x^2, \tag{1}$$

which is the equation required.

Cor.—When $y = 0$, we have $x - p = \pm ex$; therefore

$$x = \frac{p}{1 \mp e},$$

which shows that the curve cuts the axis of x at the two points A' and A, giving

$$OA' = \frac{p}{1 + e}, \quad \text{and} \quad OA = \frac{p}{1 - e}.$$

A and A' are called the **Vertices** of the ellipse, and C, the point midway between them, is called the **Centre** of the ellipse.

The distance $AA' = \frac{p}{1 - e} - \frac{p}{1 + e} = \frac{2ep}{1 - e^2}$; it is customary to denote this distance by $2a$. Therefore,

$$2a = \frac{2ep}{1 - e^2}.$$

70. Transform equation (1) of Art. 69 to A′. The formulæ for this transformation become

$$x = \frac{p}{1+e} + x', \qquad y = y',$$

which in (1) gives

$$\left(\frac{p}{1+e} + x' - p\right)^2 + y'^2 = e^2\left(\frac{p}{1+e} + x'\right)^2,$$

or $$\left(x' - \frac{ep}{1+e}\right)^2 + y'^2 = e^2\left(\frac{p}{1+e} + x'\right)^2,$$

or $$x'^2 + y'^2 - \frac{2ep}{1+e}x' = e^2\left(\frac{2p}{1-e}x' + x'^2\right);$$

therefore, $$y'^2 = 2epx' - (1-e^2)\,x'^2$$

$$= (1-e^2)\left(\frac{2pe}{1-e^2}x' - x'^2\right);$$

therefore (Art. 69, Cor.),

$$y'^2 = (1-e^2)\,(2ax' - x'^2)\,;$$

or, suppressing the accents, since the variables are general, we have,

$$y^2 = (1-e^2)\,(2ax - x^2). \tag{1}$$

Cor.—When $x = a$, equation (1) becomes

$$y^2 = (1-e^2)\,a^2 = b^2 \text{ (by putting CB} = b).$$

Therefore $$1 - e^2 = \frac{b^2}{a^2},$$

which in (1) gives $$y^2 = \frac{b^2}{a^2}(2ax - x^2). \tag{2}$$

71. Transform equation (2) of Art. 70 to C (Fig. 62). The formulæ for this transformation become

$$x = a + x', \qquad y = y';$$

which in (2) gives, after suppressing accents,

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2), \tag{1}$$

or $$a^2y^2 + b^2x^2 = a^2b^2, \tag{2}$$

which may be more symmetrically written,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \tag{3}$$

a form analogous to that of the equation of a right line (Art. 22, II).

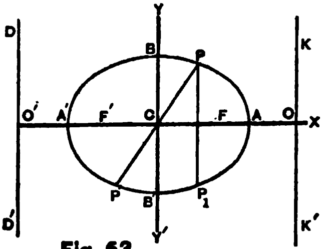

Fig. 63.

Cor. 1.* $A'F' = e \cdot A'O'$ (by definition), and

$$O'A' = \frac{p}{1+e} \text{ (Art. 69, Cor.)};$$

therefore,

$$A'F' = \frac{ep}{1+e}$$

$$= (1-e)\frac{ep}{1-e^2},$$

or $$A'F' = a(1-e), \text{ by (Art. 69, Cor.)}; \tag{4}$$

$$O'A' = \frac{p}{1+e} = \frac{ep(1-e)}{e(1-e^2)},$$

or $$O'A' = \frac{a(1-e)}{e} \text{ (Art. 69, Cor.)}. \tag{5}$$

$$F'C = A'C - A'F' = a - a(1-e),$$

or $$F'C = ae. \tag{6}$$

$$O'C = A'C + O'A' = a + \frac{a(1-e)}{e}$$

or $$O'C = \frac{a}{e}. \tag{7}$$

$$O'F' = p = O'C - F'C = \frac{a}{e} - ae,$$

or $$O'F' = p = \frac{a(1-e^2)}{e}. \tag{8}$$

Cor. 2.—When $y=0$, $x = \pm a$, which shows that the curve cuts the axis of x at two points, equally distant from the origin, and on opposite sides of it. When $x = 0$, $y = \pm b$, which shows that the curve cuts the axis of y at two points equally distant from the origin, and on opposite sides of it.

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* See Todhunter's Conic Sections, p. 145.

COR. 3.—Solving (1) for y, we get

$$y = \pm \frac{b}{a} \sqrt{a^2 - x^2},$$

which shows that for every value of x between $-a$ and $+a$ there are *two real* values of y, numerically equal, but with contrary signs; hence, for every point P on one side of the axis of x there is a point P_1 on the other side of the axis, at the same distance from it, and therefore the curve is symmetrical with respect to the axis of x. When x is $+a$ or $-a$, $y = \pm 0$, and for every value of $x > +a$ or $< -a$, the two values of y are imaginary; therefore the curve is limited in the direction of *positive* and *negative* abscissas by two tangents at A and A′.

Similarly, solving (1) for x, we get

$$x = \pm \frac{a}{b} \sqrt{b^2 - y^2},$$

which shows that for every value of y between $-b$ and $+b$ there are two values of x, numerically equal with contrary signs; hence the curve is symmetrical with respect to the axis of y, and is limited in the direction of *positive* and *negative* ordinates by two tangents at B and B′.

SCH.—Because the curve is symmetrical with respect to the line BB′, Fig. 63, it follows that if we take CF = CF′, and CO = CO′, and draw KK′ perpendicular to OO′, the point F and the line KK′ will form respectively a second focus and directrix.

AA′ is called the **Transverse** or **Major* Axis** of the ellipse, BB′ is called the **Conjugate** or **Minor* Axis** of the ellipse. The ratio e, which the distance of any point in the ellipse from the focus bears to the distance of the same point from the corresponding directrix, is called the **Eccentricity** of the ellipse.

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* Called *major* and *minor*, because, from Art. 70, Cor., $b^2 = (1 - e^2)a^2$; and $\therefore b^2 < a^2$ and $b < a$.

A **Centre** of a curve is a point which bisects every right line drawn through it to meet the curve. If x', y' satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$-x'$, $-y'$ will also satisfy it; hence, if (x', y') be any point P on the ellipse, $(-x', -y')$ will be a second point P′ on the ellipse in the opposite quadrant, such that PCP′ is a right line bisected at C; that is, every chord passing through C is bisected at C, which is therefore the centre of the ellipse.

Cor. 4.—*To find the latus rectum* (Art. 53, Cor. 3).

Make $x = \mathrm{CF} = ae$ (Cor. 1); denote the corresponding value of y by p; we have from (1),

$$p^2 = \frac{b^2}{a^2}(a^2 - a^2e^2) = b^2(1 - e^2)$$

$$= \frac{b^4}{a^2} \text{ (Art. 70, Cor.)};$$

therefore, $$2p = \frac{2b^2}{a} = \textit{latus rectum};$$

or $$2p = \frac{4b^2}{2a}.$$

Forming a proportion from this equation, we have

$$2a : 2b :: 2b : 2p.$$

That is, *the latus rectum is a third proportional to the major and minor axes.*

Since $b^2 = a^2(1 - e^2)$, (Art. 70, Cor.), we have

$$b^2 + a^2e^2 = a^2;$$

that is, $\overline{CB}^2 + \overline{CF}^2 = a^2.$

Hence, $BF = a = BF'.$

Cor. 5.—If (x', y') and (x'', y'') be any two points on the curve, we shall have, from equation (1),

$$y'^2 = \frac{b^2}{a^2}(a^2 - x'^2),$$

and $$y''^2 = \frac{b^2}{a^2}(a^2 - x''^2).$$

Hence, forming a proportion, we have

$$y'^2 : y''^2 :: (a - x')(a + x') : (a - x'')(a + x'');$$

that is, *the squares of any two ordinates to the major axis of the ellipse are to each other as the rectangles of the segments into which they divide the major axis.*

It may be proved in a similar manner that the squares of ordinates to the *minor axis* are to each other as the products of the parts into which they divide the minor axis.

Cor. 6.—A point is *outside, on,* or *inside* the ellipse, according as $a^2y^2 + b^2x^2 - a^2b^2 >, =,$ or $< 0.$

Thus, if the point is *on* the curve, as at P, its co-ordinates will satisfy the equation of the curve, giving

$$a^2y^2 + b^2x^2 - a^2b^2 = 0.$$

If the point is *outside* of the curve, as at B, its abscissa will be greater than at P, while its ordinate will be the same, giving

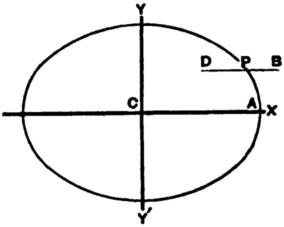

Fig. 64.

$$a^2y^2 + b^2x^2 - a^2b^2 > 0.$$

If the point is *inside* of the curve, as at D, its abscissa will be less than at P, while its ordinate will be the same, giving

$$a^2y^2 + b^2x^2 - a^2b^2 < 0.$$

COR. 7.—If $b = a$, (2) becomes $x^2 + y^2 = a^2$, which is the equation of a circle. Hence, the ellipse becomes a circle when its axes are made equal to each other.

72. *To find the distance of any point in the ellipse from the focus, in terms of the abscissa of the point.*

From the figure we have

$$\overline{FP}^2 = (ae - x)^2 + y^2$$

$$= (ae - x)^2 + b^2 - \frac{b^2}{a^2}x^2$$

(Art. 71, Eq. 1),

$$= a^2 - 2aex + e^2x^2$$

(since $a^2e^2 + b^2 = a^2$); therefore,

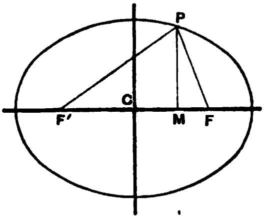

Fig. 65.

$$FP = (a - ex).$$

We take only the *positive* value, since it is the *absolute* distance of P from F we are considering, and not the *direction.*

In like manner we find, by writing $-ae$ for ae,

$$\overline{F'P}^2 = (a + ex)^2 + y^2$$

$$= a^2 + 2aex + e^2x^2;$$

therefore, $F'P = a + ex.$

Hence, $FP + F'P = 2a;$

or, *the sum of the distances of any point in an ellipse from the foci is equal to the major axis.*

COR.—This result furnishes two other methods of constructing an ellipse, having given the axes.

I. With B as a centre and CA as a radius, describe an arc cutting AA′ at F and F′; these points are the foci. Now with F′ as a centre

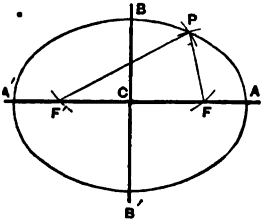

Fig. 66.

and a radius greater than A′F′ and less than AF, describe an arc; then with F as a centre and the remainder of the major axis as a radius, describe another arc cutting the first at the point P; this will be a point of the curve, since $F'P + FP = 2a$. In the same way, any number of points may be found; joining these points by a curve, it will be the required ellipse.

II. Take a string equal in length to AA′, and fix the two ends at F and F′; then press a pencil-point P against the string, and move it around F and F′, keeping the string tight; the pencil will describe an ellipse, since in every position of it we shall have $F'P + FP = 2a$.

73. *If a circle be described on the major axis as a diameter, then any ordinate of the ellipse is to the corresponding ordinate of the circumscribed circle as the semi-minor axis is to the semi-major axis.*

Produce the ordinate MP of the ellipse to meet the circumscribed circle at P′. The points P and P′ are called **Corresponding Points**; the ordinates MP and MP′ are called **Corresponding Ordinates**. Denote the ordinate MP by y and MP′ by y'. Then the equations of the ellipse and circle, referred to the centre C, are respectively,

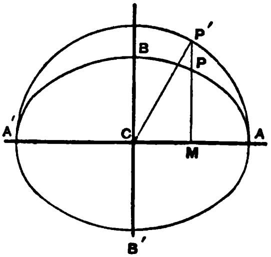

Fig. 67.

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2). \qquad (1)$$

$$y'^2 = a^2 - x^2. \qquad (2)$$

Making the x of (1) and (2) identical, the values of y and y' will represent corresponding ordinates. Divide (1) by (2), and extract the square root; and we have

$$\frac{y}{y'} = \frac{b}{a}, \tag{3}$$

and forming a proportion, we have

$$y : y' :: b : a.$$

Cor. 1.—In the same way it may be proved that, if a circle be described on the *minor axis* of an ellipse as a diameter, *any abscissa of the ellipse is to the corresponding abscissa of the inscribed circle as the semi-major axis is to the semi-minor axis.* [Let the student give the proof.]

Cor. 2.—Join P′ with C, the centre of the ellipse; denote ACP′ by ϕ, and let (x, y) be the point P. Then we have, from Fig. 67,

$$x = \text{CM} = \text{CP}' \cos \text{ACP}' = a \cos \phi, \tag{4}$$

and $$y = \text{MP}' = \text{CP}' \sin \text{ACP}' = a \sin \phi, \tag{5}$$

which in (3) gives, $$y = b \sin \phi. \tag{6}$$

These values of x and y enable us *to express the position of any point on an ellipse in terms of a single variable.*

The angle ACP′ is called the **Eccentric Angle** of the point P, and the circle described on the major axis of an ellipse as a diameter is sometimes called the **Auxiliary Circle.**

74. *To find the equation of the tangent to an ellipse at any point.* (See Def., Art. 42.)

Let (x', y') and (x'', y'') be any two points on the curve. The equation of the secant through these points (Art. 26) is

$$y - y' = \frac{y' - y''}{x' - x''}(x - x'). \tag{1}$$

Since (x', y') and (x'', y'') are on the ellipse, they will satisfy the equation of the ellipse, giving us

$$a^2y'^2 + b^2x'^2 = a^2b^2. \tag{2}$$

$$a^2y''^2 + b^2x''^2 = a^2b^2. \tag{3}$$

Subtracting (3) from (2), factoring, and dividing, we have

$$\frac{y'-y''}{x'-x''} = -\frac{b^2}{a^2}\left(\frac{x'+x''}{y'+y''}\right),$$

which in (1) gives

$$y-y' = -\frac{b^2}{a^2}\cdot\frac{x'+x''}{y'+y''}(x-x'). \qquad (4)$$

When the points become consecutive, $x''=x'$ and $y''=y'$; hence (4) becomes

$$y-y' = -\frac{b^2x'}{a^2y'}(x-x'). \qquad (5)$$

Clearing of fractions, transposing, and substituting for $a^2y'^2 + b^2x'^2$ its value, a^2b^2 [see (2)], we have

$$a^2yy' + b^2xx' = a^2b^2, \qquad (6)$$

which is the required equation of the tangent.

This equation may be written

$$y = -\frac{b^2x'}{a^2y'}x + \frac{b^2}{y'}, \qquad (7)$$

in which $-\frac{b^2x'}{a^2y'}$ is the tangent of the angle which the tangent line at the point (x', y') makes with the axis of x.

COR. 1.—We may write the equation of the tangent in terms of the tangent of the angle which the tangent makes with the major axis.

Since the tangent is a right line, its equation may be written in the form

$$y = mx + n. \qquad (8)$$

Writing (6) and (8) in the form

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1, \qquad (9)$$

$$-\frac{m}{n}x + \frac{y}{n} = 1, \qquad (10)$$

and comparing (9) and (10), since they are to represent the same line, we have

$$\frac{x'}{a^2} = -\frac{m}{n} \qquad \text{and} \qquad \frac{y'}{b^2} = \frac{1}{n};$$

and therefore, since $\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = 1$ (Art. 71, Equation 3),

we have $$\left(\frac{am}{n}\right)^2 + \left(\frac{b}{n}\right)^2 = 1,$$

which gives $$n^2 = a^2m^2 + b^2,$$

or $$n = \pm\sqrt{a^2m^2 + b^2};$$

which in (8) gives $$y = mx \pm \sqrt{a^2m^2 + b^2}, \tag{11}$$

the equation required. The double sign shows that *two* tangents may be drawn at the same angle to the axis of x.

Cor. 2.—To find the point in which the tangent cuts the axis of x, make $y = 0$ in (6), and get

$$x = \frac{a^2}{x'} = \text{CT}.$$

The *subtangent* MT (Art. 43),

$$= \text{CT} - \text{CM} = \frac{a^2}{x'} - x'$$

$$= \frac{a^2 - x'^2}{x'}.$$

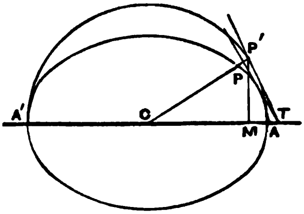

Fig. 68.

Sch.—This expression for the subtangent is independent of b and y'; hence, all ellipses having the same major axis, will have the same subtangent for any given abscissa of contact.

This principle enables us to draw a tangent to an ellipse at a given point. Let P be any point on the ellipse, Fig. 68. On AA′ describe the circle; and at P′, where the ordinate of the ellipse at P, prolonged, meets the circle, draw the tangent P′T; join TP, and it will be the tangent required at P.

75. *To find the equation of the normal at any point of an ellipse.*

Let (x', y') be the point; the equation of the *tangent* at (x', y') is

$$y = -\frac{b^2x'}{a^2y'}x + \frac{b^2}{y'}. \qquad (1)$$

The equation of a right line through (x', y') perpendicular to (1) is, by Art. 27, Cor. 2,

$$y - y' = \frac{a^2y'}{b^2x'}(x - x'), \qquad (2)$$

which is the required equation of the normal at the point (x', y').

Cor. 1.—To find the point in which the normal at (x', y') cuts the axis of x, we make $y = 0$ in (2), and get, after reduction,

$$x = \frac{a^2 - b^2}{a^2}x' = \text{CN}$$

$$= e^2x' \text{ (Art. 70, Cor.)}$$

The *subnormal* NM (Art. 43),

$$= \text{CM} - \text{CN} = x' - \frac{a^2 - b^2}{a^2}x'$$

$$= \frac{b^2}{a^2}x'.$$

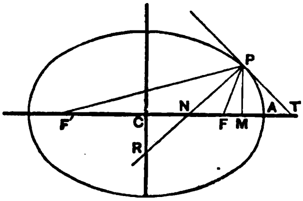

Fig. 69.

Sch.—The expression $\text{CN} = e^2x'$ enables us to draw a normal, either through a given point on the major axis or at a given point on the curve. For, in the former case, we have given CN to find $\text{CM} = x'$, and in the latter case CM is given, to find $\text{CN} = x$.

Cor. 2.—By Art. 71, Cor. 1, we have

$$\text{F'C} = \text{FC} = ae;$$

therefore, by Cor. 1 of the present Art., we have.

$$F'N = e(a + ex')$$

and $$FN = e(a - ex').$$

Hence, $F'N : FN :: a+ex' : a-ex' :: F'P : FP$ (Art. 72).

That is, *the normal of an ellipse cuts the distance between the foci in segments proportional to the adjacent focal radii of contact; and therefore it bisects the angle between these focal radii.*

Cor. 3.—*Since the tangent is perpendicular to the normal, it makes equal angles with the focal radii to the point of contact.*

Sch. 2.—This principle affords a method of drawing a tangent to an ellipse at a given point.

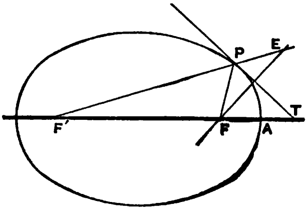

Fig. 70.

Let P be the given point, and FP and F'P the focal radii to the point of contact. Produce F'P to E, making PE = PF, and draw FE. Draw PT perpendicular to FE, and it will be the tangent required.

76. *To find the locus of the intersection of the tangent at any point with the perpendicular on it from either focus.*

Let (x', y') be any point of the curve. By Art. 74, Cor. 1, the equation of the tangent at (x', y') is

$$y = mx + \sqrt{a^2m^2 + b^2}. \qquad (1)$$

The equation of the line through the focus F perpendicular to (1) is (Art. 27, Cor. 2),

$$y = -\frac{1}{m}(x - ae). \qquad (2)$$

(1) becomes $$y - mx = \sqrt{a^2m^2 + b^2}. \qquad (3)$$

(2) becomes $$my + x = ae. \qquad (4)$$

$(3)^2 + (4)^2$ gives, after dividing by $1 + m^2$,

$$x^2 + y^2 = a^2,$$

as the equation of the required locus, which is therefore a circle described on the major axis as a diameter.

77. *To find the co-ordinates of the point of contact of a tangent to an ellipse from a fixed point.*

Let (x', y') be the required point of contact, and (x'', y'') the fixed point through which the tangent passes

Since (x', y') is on the ellipse, we have

$$a^2y'^2 + b^2x'^2 = a^2b^2. \qquad (1)$$

The equation of the tangent at (x', y') is

$$a^2yy' + b^2xx' = a^2b^2,$$

and since this tangent passes through (x'', y''), we have

$$a^2y''y' + b^2x''x' = a^2b^2. \qquad (2)$$

Solving (1) and (2) for x' and y', we find

$$x' = \frac{a^2b^2x'' \pm a^2y''\sqrt{a^2y''^2 + b^2x''^2 - a^2b^2}}{a^2y''^2 + b^2x''^2},$$

$$y' = \frac{a^2b^2y'' \mp b^2x''\sqrt{a^2y''^2 + b^2x''^2 - a^2b^2}}{a^2y''^2 + b^2x''^2}.$$

These values indicate that from any fixed point *two* tangents can be drawn to an ellipse, *real, coincident,* or *imaginary,* according as $a^2y''^2 + b^2x''^2 - a^2b^2 >, =, < 0$; that is, according as the point (x'', y'') is *without, on,* or *within* the curve. (See Art. 71, Cor. 6.)

The line joining these two *real* points of contact is called the **Chord of Contact.**

78. *Tangents are drawn to an ellipse from a given external point; to find the equation of the chord of contact.*

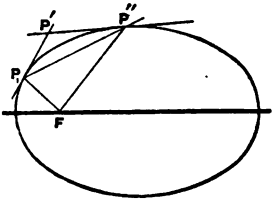

Fig. 71.

Let (x', y') be the external point P'; (x'', y'') and (x_1, y_1) the two points P'' and P₁ where the tangents meet the ellipse. Then P''P₁ will be the chord of contact whose equation is required.

The equation of the tangent at P'' is (Art. 74),

$$a^2yy'' + b^2xx'' = a^2b^2; \qquad (1)$$

and since this tangent passes through P', we have

$$a^2y'y'' + b^2x'x'' = a^2b^2. \qquad (2)$$

Also, the equation of the tangent at P₁ is

$$a^2yy_1 + b^2xx_1 = a^2b^2; \qquad (3)$$

and since this tangent passes through P', we have

$$a^2y'y_1 + b^2x'x_1 = a^2b^2. \qquad (4)$$

Hence, it follows that the equation of the chord of contact P''P₁ is

$$a^2yy' + b^2xx' = a^2b^2. \qquad (5)$$

For (5) is the equation of *some right* line (Art. 23); and this line passes through P'', for (5) is satisfied by the coordinates of P'' $(x = x'', y = y'')$, as we see from (2). Also, the line passes through P₁, for (5) is satisfied by the co-ordinates of P₁ $(x = x_1, y = y_1)$, as we see from (4). Hence, (5) is the required equation.

79. *Through any fixed point a chord is drawn to an ellipse, and tangents to the ellipse are drawn at the extremities of the chord; to find the equation of the locus of the intersection of the tangents, when the chord is turned about the fixed point.*

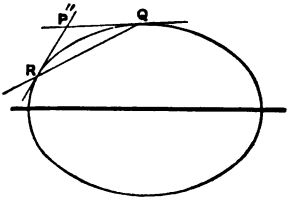

Fig. 72.

Let (x', y') be the fixed point P′ through which the chord passes; and (x'', y'') the point P″ in which the two tangents, drawn at the extremities Q and R of the chord, intersect.

The equation of the chord of contact (Art. 78) is

$$a^2yy'' + b^2xx'' = a^2b^2. \qquad (1)$$

Since this chord passes through P′, we have

$$a^2y'y'' + b^2x'x'' = a^2b^2. \qquad (2)$$

Now, if the chord turns about the fixed point (x', y'), the intersection (x'', y'') of the corresponding tangents is *any* point in the required locus; that is, the co-ordinates x'', y'' of any point in the required locus satisfy the equation

$$a^2yy' + b^2xx' = a^2b^2, \qquad (3)$$

which is therefore the equation required, and the locus is a right line.

SCH.—The line $a^2yy' + b^2xx' = a^2b^2$ is called the **Polar** of the point (x', y') with regard to the ellipse

$$a^2y^2 + b^2x^2 = a^2b^2,$$

and the point (x', y') is called the **Pole** of the line.

It will be seen that if (x', y') be *any point whatever*, the equation

$$a^2yy' + b^2xx' = a^2b^2$$

represents the locus of the intersection of the tangents at the extremities of the chord through (x', y').

The statements in Art. 49 with respect to the circle may all be applied to the ellipse as they were to the parabola (Art. 61). By the same reasoning, then, as in Arts. 49 and 61, we learn that the tangent and chord of contact in the ellipse are particular cases of the locus just described, due to bringing the point (x', y') *on* the curve, or *outside* of it.

80. *To find the equation of any diameter.* (See Def., Art. 62.)

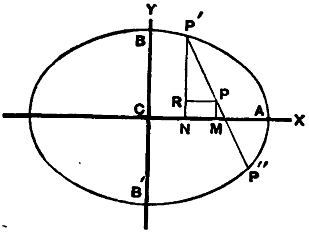

Fig. 73.

Let (x, y) be the middle point P of the chord P′P″; (x', y') the point P′ or P″; θ the inclination of P′P″ to the axis of x; r the length of PP′, half the chord P′P″. Then

$$x' = \text{CM} - \text{NM} = x + r\cos\theta,$$

$$y' = \text{NR} + \text{RP}' = y + r\sin\theta.$$

Now, as P′ is on the curve, its co-ordinates x', y' will satisfy the equation of the curve, $a^2y^2 + b^2x^2 = a^2b^2$, giving

$$a^2(y + r\sin\theta)^2 + b^2(x + r\cos\theta)^2 = a^2b^2,$$

or,
$$\left\{\begin{array}{l} \quad (a^2\sin^2\theta + b^2\cos^2\theta)\,r^2 \\ + 2(a^2y\sin\theta + b^2x\cos\theta)\,r \\ + (a^2y^2 + b^2x^2 - a^2b^2) \end{array}\right\} = 0, \qquad (1)$$

from which quadratic we can determine the two values of r. But as (x, y) is the middle point P of the chord, the two values of r are numerically equal, with contrary signs; therefore (from Algebra), the coefficient of the first power of r vanishes, giving us,

$$a^2y\sin\theta + b^2x\cos\theta = 0,$$

or
$$y = -\frac{b^2}{a^2}\cot\theta \cdot x, \qquad (2)$$

which represents the locus of the middle point P of the chord P′P″; and is therefore the required equation of any diameter.

Since a^2 and b^2 are fixed for any given ellipse, and θ is constant for any given system of parallel chords, (2) is the equation of a right line passing through the origin (Art. 22, I, Cor. 2), that is, through the centre of the ellipse. Hence, every diameter of the ellipse passes through its centre. By giving to θ a suitable value, equation (2) may be made to represent *any* right line passing through the centre. Hence it follows that every right line that passes through the centre of an ellipse is a diameter; that is, it bisects some system of parallel chords.

COR.—Let $\theta' =$ the inclination to the major axis of the diameter that bisects all the chords inclined at an angle θ; then clearly we have

$$\tan\theta' = \frac{y}{x},$$

which in (2) gives $$\tan\theta' = -\frac{b^2}{a^2}\cot\theta;$$

therefore, $$\tan\theta\tan\theta' = -\frac{b^2}{a^2}.$$

Hence, if θ and θ' be the angles which a system of parallel chords and their diameter respectively make with the axis of x, we have the relation

$$\tan\theta\tan\theta' = -\frac{b^2}{a^2}.$$

SCH.—To draw a diameter to an ellipse, draw any two parallel chords, and bisect them; the line passing through the points of bisection is a diameter. The intersection of two diameters will be the centre of the ellipse.

81. *If one diameter of an ellipse bisects all chords parallel to a second diameter, the second will bisect all chords parallel to the first.*

Let θ and θ' be the respective inclinations of any two diameters to the major axis. Then the condition that the first diameter shall bisect all chords parallel to the second diameter (Art. 80, Cor.) is

$$\tan\theta\tan\theta' = -\frac{b^2}{a^2}.$$

But this is also the condition that the second diameter bisects all chords parallel to the first.

SCH.—Two diameters are **Conjugate** when each bisects all chords parallel to the other.

Because the conjugate of any diameter is parallel to the chords which the diameter bisects, therefore the inclinations of two conjugates must be connected in the same way as those of a diameter and its bisected chords. Hence, if θ and θ' are the inclinations, the *equation of condition for conjugate diameters* is (Art. 80, Cor.),

$$\tan\theta \tan\theta' = -\frac{b^2}{a^2}.$$

Since this condition shows that the tangents of inclination of any two conjugate diameters have opposite signs, it indicates that one of the two conjugates makes an *acute* angle with the major axis, and the other an *obtuse* angle. The minor axis makes a *right angle* with the major axis; therefore, *conjugate diameters of an ellipse lie on opposite sides of the minor axis.*

82. *The tangent at either extremity of any diameter is parallel to its conjugate diameter.*

Let (x', y') be either extremity of any diameter; θ the inclination of its conjugate to the major axis. Since (x', y') is on the diameter, its co-ordinates will satisfy its equation, giving us (Art. 80),

$$y' = -\frac{b^2}{a^2}\cot\theta \cdot x';$$

therefore,
$$\tan\theta = -\frac{b^2x'}{a^2y'}. \qquad (1)$$

But, Art. 74, Eq. (7), the equation of the tangent at (x', y') is

$$y = -\frac{b^2x'}{a^2y'}x + \frac{b^2}{y'}; \qquad (2)$$

therefore, the tangent at the extremity of any diameter is parallel to its conjugate diameter.

83. *Given the co-ordinates x', y' of one extremity of a diameter, to find the co-ordinates x'', y'' of either extremity of the conjugate diameter.*

Since the conjugate diameter passes through the origin, and is parallel to the tangent at (x', y'), by Art. 82, therefore its equation (Art. 74) is

$$y = -\frac{b^2x'}{a^2y'}x. \tag{1}$$

Substitute this value of y in the equation of the ellipse,

$$a^2y^2 + b^2x^2 = a^2b^2,$$

and, after reducing, we obtain

$$x'' = \pm\frac{a}{b}y'; \tag{2}$$

and from (1) we have $y'' = \mp\frac{b}{a}x'$. (3)

The upper sign in each of these values belongs to the extremity at the *right* of the origin, and the lower sign to the left extremity.

84. *To express the lengths of a semi-diameter (a'), and its conjugate (b'), in terms of the abscissa of the extremity of the diameter.*

Let (x', y') and (x'', y'') be the extremities of the diameters (a') and (b'); then we have

$$a'^2 = x'^2 + y'^2 = x'^2 + \frac{b^2}{a^2}(a^2 - x'^2) \text{ (Art. 71)}$$

$$= b^2 + \frac{a^2 - b^2}{a^2}x'^2 = b^2 + e^2x'^2 \text{ (Art. 71, Cor. 4).} \tag{1}$$

Also, $b'^2 = x''^2 + y''^2 = \frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2$ (Art. 83)

$$= a^2 - x'^2 + \frac{b^2}{a^2}x'^2 \text{ (Art. 71);}$$

hence, $b'^2 = a^2 - e^2x'^2$. (2)

COR.—Adding (1) and (2), we get

$$a'^2 + b'^2 = a^2 + b^2;$$

that is, the sum of the squares of any pair of conjugate diameters is equal to the sum of the squares of the axes. [See Salmon's Conic Sections, p. 163.]

85. *To find the length of the perpendicular from the centre to the tangent at any point.*

Let (x', y') be the point, and p the perpendicular. The equation of the tangent at (x', y') is (Art. 74),

$$a^2yy' + b^2xx' = a^2b^2.$$

Therefore (Art. 24),

$$p = \frac{a^2b^2}{\sqrt{a^4y'^2 + b^4x'^2}} = \frac{ab}{\sqrt{\frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2}} = \frac{ab}{b'}. \quad \text{(Art. 84.)}$$

86. *To find the angle between any pair of conjugate diameters.*

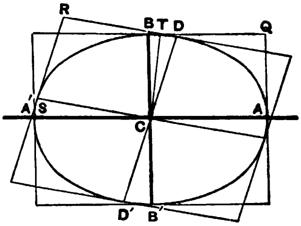

Fig. 74.

Let ϕ be the required angle $=$ SCD in the figure. The angle between the two conjugate diameters is equal to the angle between either diameter and the tangent parallel to the other.

From the figure,

$$\sin\phi = \sin \text{SCD} = \sin \text{CDR} = \frac{\text{CT}}{\text{CD}} = \frac{p}{a'} = \frac{ab}{a'b'}. \quad \text{(Art. 85.)}$$

That is, $$\sin\phi = \frac{ab}{a'b'}. \qquad (1)$$

Cor.—Clearing (1) of fractions, we have

$$a'b' \sin\phi = ab, \tag{2}$$

which shows that the parallelogram CDRS is equal to the rectangle CAQB. Hence, *the area of the parallelogram whose sides touch the ellipse at the ends of any pair of conjugate diameters is constant, and equal to the rectangle of the axes.*

Sch.—Since the sum of the squares of any pair of conjugate diameters, (a') and (b'), of an ellipse is constant (Art. 84, Cor.), the rectangle $a'b'$ is the greatest when $a' = b'$,* and therefore $\sin\phi$ is the least; or, *the obtuse angle ϕ is the greatest when the conjugate diameters are of equal lengths.* These diameters are called **Equi-conjugate** diameters. Their length is found by making $a' = b'$ in the equation of Art. 84, Cor., giving

$$a'^2 = \tfrac{1}{2}(a^2 + b^2) = b'^2;$$

therefore,

$$a' = b' = \tfrac{1}{2}\sqrt{2} \cdot \sqrt{a^2 + b^2}.$$

Therefore, by (1), $\sin\phi = \dfrac{ab}{a'b'} = \dfrac{2ab}{a^2 + b^2}$.

87. *To prove that the eccentric angles of the vertices of two conjugate diameters differ from each other by 90°.*

Let ϕ be the eccentric angle corresponding to the point D (x', y'), and ϕ' the eccentric angle corresponding to the point S (x'', y''), in Fig. 74.

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* Let $a + b = c, \quad (1) \qquad a - b = d, \quad (2)$

$$\therefore \quad a = \frac{c+d}{2}, \qquad b = \frac{c-d}{2},$$

and $ab \left(= \dfrac{c^2 - d^2}{4}\right)$ is greatest, since c is constant, when $d = 0$, *i. e.*, from (2), when $a = b$. In the present case,

$$c = a'^2 + b'^2, \qquad d = a'^2 - b'^2 = 0.$$

$\therefore$ $a'^2 b'^2$ is greatest when $a'^2 = b'^2$; and inasmuch as we are concerned only with numerical values of a' and b', $a'b'$ is greatest when $a' = b'$.

Then (Art. 73, Cor. 2), we have

$$x' = a \cos \phi, \tag{1}$$

$$y' = b \sin \phi. \tag{2}$$

Also (Art. 83), $x'' = -\frac{a}{b}y' = a \cos\phi'$ (Art. 73, Cor. 2)

and $y'' = \frac{b}{a}x' = b \sin \phi'$;

or, $$y' = -b \cos \phi', \tag{3}$$

and $$x' = a \sin \phi'. \tag{4}$$

Divide (2) by (1), and get

$$\frac{y'}{x'} = \frac{b}{a} \tan \phi. \tag{5}$$

Divide (4) by (3), and get

$$\frac{x'}{y'} = -\frac{a}{b} \tan \phi'. \tag{6}$$

Multiply (5) by (6), and get

$$\tan \phi \tan \phi' = -1,$$

or, $$\tan \phi \tan \phi' + 1 = 0.$$

Therefore (Art. 27, Cor. 1), the two angles ϕ and ϕ' differ from each other by 90°.

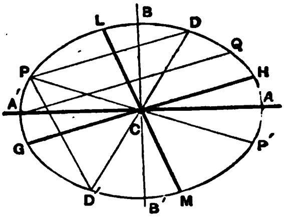

Fig. 75.

88. Two chords which join the extremities of any diameter to any point on the ellipse are called **Supplemental Chords.** If that diameter be the major axis, the chords are called **Principal** supplemental chords.

Thus, DP and D'P are supplemental with respect to the diameter DD'; AQ and A'Q with respect to the major axis.

89. *If a chord and diameter of an ellipse are parallel, the supplemental chord and the conjugate diameter are parallel.*

Let DD′ (Fig. 75) be a diameter of the ellipse; PD and PD′ two supplemental chords, the first parallel to the diameter GH. Let (x', y') be the point D, and therefore $(-x', -y')$ will be the point D′. Let ϕ be the inclination of the chord DP to the major axis, and ϕ' the inclination of the supplemental chord D′P. Then the equation of DP (Art. 25) is

$$y - y' = \tan\phi\,(x - x'), \tag{1}$$

and the equation of D′P is

$$y + y' = \tan\phi'\,(x + x'). \tag{2}$$

Since these lines are to intersect at the point P (x, y), we combine (1) and (2), and get

$$y^2 - y'^2 = \tan\phi\tan\phi'\,(x^2 - x'^2). \tag{3}$$

And since (x, y) and (x', y') are points on the ellipse, their co-ordinates must satisfy the equation of the ellipse, giving

$$a^2y^2 + b^2x^2 = a^2b^2,$$

and

$$a^2y'^2 + b^2x'^2 = a^2b^2;$$

from which we get $y^2 - y'^2 = -\dfrac{b^2}{a^2}(x^2 - x'^2)$;

which in (3) gives $\tan\phi\tan\phi' = -\dfrac{b^2}{a^2}$, which is the condition that two chords shall be supplemental.

But (Art. 81, Sch.) the condition that two diameters are conjugate to each other is

$$\tan\theta\tan\theta' = -\frac{b^2}{a^2};$$

therefore, if $\phi = \theta$, we have $\phi' = \theta'$. But, as the chord PD and diameter GH are parallel, by hypothesis, ϕ does equal θ (if we call θ the inclination of GH); therefore, $\phi' = \theta'$; or, the supplemental chord PD′ and the conjugate diameter LM are parallel.

90. *To find the equation of the ellipse referred to any pair of conjugate diameters.*

To do this we must transform the equation of the ellipse

$$a^2y^2 + b^2x^2 = a^2b^2, \tag{1}$$

from rectangular to oblique axes, having the same origin.

Let DD′ and SS′ be two conjugate diameters. Take CD for the new axis of x, and CS for the new axis of y. Denote the angles ACD and ACS by θ and θ' respectively. Let x, y be the co-ordinates of any point P of the ellipse referred to the old axes, and x', y' the co-ordinates of the same point referred to the new axes.

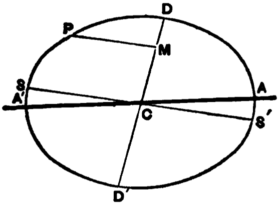

Fig. 76.

The formulæ for transformation (Art. 35, Cor. 1) are,

$$x = x' \cos\theta + y' \cos\theta',$$

$$y = x' \sin\theta + y' \sin\theta',$$

since m and n are 0.

Squaring, substituting in (1), and arranging, we have

$$\left.\begin{array}{r} (a^2\sin^2\theta + b^2\cos^2\theta)\,x'^2 \\ +\,(a^2\sin^2\theta' + b^2\cos^2\theta')\,y'^2 \\ +\,2\,(a^2\sin\theta\sin\theta' + b^2\cos\theta\cos\theta')\,x'y' \end{array}\right\} = a^2b^2, \tag{2}$$

which is the equation of the ellipse when the oblique axes make *any angles* θ and θ' with the major axis.

But since the new axes CD and CS are conjugate diameters, we have (Art. 81, Sch.),

$$\tan\theta\tan\theta' = -\frac{b^2}{a^2};$$

or, $$a^2\sin\theta\sin\theta' + b^2\cos\theta\cos\theta' = 0;$$

hence the coefficient of $x'y'$ in (2) vanishes, and it becomes,

$$(a^2 \sin^2\theta + b^2 \cos^2\theta)\, x'^2 + (a^2 \sin^2\theta' + b^2 \cos^2\theta')\, y'^2 = a^2b^2, \quad (3)$$

which is the equation of the ellipse referred to any two conjugate diameters.

In this equation, the coefficients are still in terms of the axes of the ellipse; we may obtain the equation in terms of the conjugate diameters lying on the new axes; thus:

If in (3) we make $y' = 0$, and represent CD by a', we have

$$x'^2 = \frac{a^2b^2}{a^2 \sin^2\theta + b^2 \cos^2\theta} = a'^2. \quad (4)$$

Also, if in (3) we make $x' = 0$, and represent CS by b', we have

$$y'^2 = \frac{a^2b^2}{a^2 \sin^2\theta' + b^2 \cos^2\theta'} = b'^2. \quad (5)$$

From (4) we get $$a^2 \sin^2\theta + b^2 \cos^2\theta = \frac{a^2b^2}{a'^2}. \quad (6)$$

From (5) we get $$a^2 \sin^2\theta' + b^2 \cos^2\theta' = \frac{a^2b^2}{b'^2}. \quad (7)$$

Substituting (6) and (7) in (3), dividing by a^2b^2, and omitting the accents from the variables, we have

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1, \quad (8)$$

or, $$a'^2y^2 + b'^2x^2 = a'^2b'^2, \quad (9)$$

which is the required equation, and is of the same form as when referred to the major and minor axes (Art. 71).

91. *To find the equation of a tangent to the ellipse referred to any pair of conjugate diameters.*

The equation of a right line referred to oblique axes is of the same form (Art. 22, IV) as when referred to rectangular axes; also, the equation of the ellipse, referred to any pair of conjugate diameters, is of the same form (Art. 90), as when referred to the axes of the ellipse. Hence, the investigation of Art. 74 will apply without any change to the equation $a'^2y^2 + b'^2x^2 = a^2b^2$, giving us the required equation,

$$a'^2yy' + b'^2xx' = a'^2b'^2. \tag{1}$$

Cor.—To find where the tangent cuts the axis of x, make $y = 0$ in (1), and get

$$x = \frac{a'^2}{x'}.$$

92. *To prove that tangents at the extremities of any chord of an ellipse meet on the diameter which bisects that chord.*

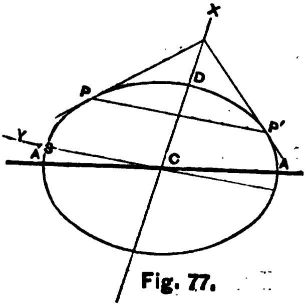

Fig. 77.

Take the diameter CD, which bisects the chord PP', for the axis of x, and the conjugate diameter CS for the axis of y.

Let (x', y') be the point P; then $(x', -y')$ will be the point P'.

The equation of the tangent at P (Art. 91) is

$$a'^2yy' + b'^2xx' = a'^2b'^2. \tag{1}$$

The equation of the tangent at P' is

$$-a'^2yy' + b'^2xx' = a'^2b'^2. \tag{2}$$

By Art. 91, Cor., both tangents cut the axis of x at the point $\left(\frac{a'^2}{x'}, 0\right)$, which proves the proposition.

93. *If tangents are drawn at the extremities of any focal chord of an ellipse:*

I. *The tangents will intersect on the corresponding directrix.*

II. *The line drawn from the point of intersection of the tangents to the focus will be perpendicular to the focal chord.*

I. If the tangents to an ellipse meet at the point (x', y'), the equation of the chord of contact (Art. 78) is

$$a^2yy' + b^2xx' = a^2b^2.$$

If the chord passes through the right-hand focus, its coordinates ($x = ae$, $y = 0$) must satisfy this equation, giving

$$b^2aex' = a^2b^2;$$

therefore

$$x' = \frac{a}{e}; \tag{1}$$

that is, the point of intersection of the tangents is on the corresponding directrix (Art. 71, Cor. 1), showing that the directrix is the polar of the focus. (Art. 79, Sch.)

II. The equation of the right line passing through the right-hand focus and the point (x', y') is, by (Art. 26),

$$y = \frac{y'}{x' - ae}(x - ae). \tag{2}$$

From (1), $x' = \frac{a}{e}$, which in (2) gives

$$y = \frac{y'e}{a - ae^2}(x - ae)$$

$$= \frac{aey'}{b^2}(x - ae). \text{ (Art. 70, Cor.)} \tag{3}$$

The equation of the chord of contact (Art. 78) is

$$y = -\frac{b^2x'}{a^2y'}x + \frac{b^2}{y'};$$

which becomes, for the focal chord [since $x' = \frac{a}{e}$, from (1)],

$$y = -\frac{b^2}{aey'}x + \frac{b^2}{y'}, \tag{4}$$

which is perpendicular to (3), by Art. 27, Cor. 1.

94. *Find the locus of the point of intersection of two tangents at right angles to each other.*

The equation of any tangent to the ellipse, by Art. 74, Cor. 1, is

$$y = mx + \sqrt{a^2m^2 + b^2}. \qquad (1)$$

The equation of the tangent at right angles to (1) is

$$y = -\frac{1}{m}x + \sqrt{\frac{a^2}{m^2} + b^2}. \qquad (2)$$

Clearing (2) of fractions, and transposing in both (1) and (2), we get,

from (1), $$y - mx = \sqrt{a^2m^2 + b^2}, \qquad (3)$$

and from (2), $$ym + x = \sqrt{a^2 + b^2m^2}. \qquad (4)$$

Adding the squares of (3) and (4) together, and dividing by the factor $(1 + m^2)$, we get

$$x^2 + y^2 = a^2 + b^2, \qquad (5)$$

which is the locus required. Hence, the locus is a circle with its centre at C, and $\sqrt{a^2 + b^2}$ for its radius. [See O'Brien's Co-ordinate Geometry, p. 118.]

95. *The rectangle of the focal perpendiculars upon any tangent is constant, and equal to the square of the semi-minor axis.*

Let p and p' be the perpendiculars, and b the semi-minor axis.

The equation of the tangent at any point (x', y') is

$$a^2yy' + b^2xx' = a^2b^2.$$

By Art. 24, we have

$$p = -\frac{b^2x'ae - a^2b^2}{\sqrt{a^4y'^2 + b^4x'^2}} = \frac{b\,(a - ex')}{\sqrt{\frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2}}$$

$$= \frac{b}{b'}(a - ex'). \quad \text{(Art. 84.)}$$

arly, $$p' = \frac{b}{b'}(a + ex').$$

ce,

$$pp' = \frac{b^2}{b'^2}(a^2 - e^2x'^2) = b^2 \quad (\text{since } a^2 - e^2x'^2 = b'^2, \text{ Art. 84}).$$

96. *To find the polar equation of the ellipse, the focus being the pole.*

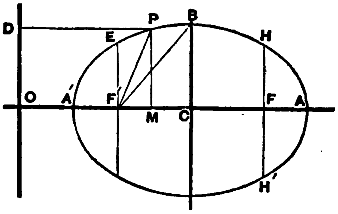

Fig. 78.

Let $F'P = r$; $AF'P = \theta$; then, by definition, Art. 68, we have,

$$\begin{aligned} F'P &= e \cdot PD \\ &= e(OF' + F'M) \\ &= e \cdot OF' + e \cdot F'M \\ &= a(1 - e^2) + e \cdot F'P \cos AF'P \end{aligned}$$

$$\left[\text{since } OF' = \frac{a(1 - e^2)}{e}, \text{ by Art. 71, Cor. 1}\right];$$

or $$r = a(1 - e^2) + er\cos\theta;$$

therefore, $$r = \frac{a(1 - e^2)}{1 - e\cos\theta}, \tag{1}$$

which is the required equation, the pole being at the left-hand focus.

Cor.—When $\theta = 0$, $r = \dfrac{a(1 - e^2)}{1 - e} = a + ae$; which makes $F'A = a + ae$, as it should do. (Art. 71, Cor. 1.)

For the point B, at the extremity of the minor axis,

$$\cos\theta = \frac{F'C}{F'B} = \frac{ae}{r};$$

which substituted for $\cos\theta$ in (1), gives

$$r = \frac{a(1-e^2)}{1-\frac{ae^2}{r}};$$

therefore, $r = a;$

that is, $F'B = a$, as it should (Art. 71, Cor. 4).

When $\theta = 90°$, $r = \frac{a(1-e^2)}{1} = \frac{b^2}{a}$ (by Art. 70, Cor.), as it should; that is, F′E, the *semi latus rectum* $= \frac{b^2}{a}$ (which agrees with Art. 71, Cor. 4).

When $\theta = 180°$, $r = \frac{a(1-e^2)}{1+e} = a - ae$, which is the value of A′F′, as it should be (Art. 71, Cor. 1).

97. *To find the polar equation of the ellipse when the pole is at the centre.*

The formulæ for passing from a rectangular to a polar system (Art. 36), are

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Substituting these values of x and y in the equation

$$a^2y^2 + b^2x^2 = a^2b^2,$$

and solving for r^2 we find

$$r^2 = \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta} = \frac{b^2}{1-\frac{a^2-b^2}{a^2}\cos^2\theta};$$

or $$r^2 = \frac{b^2}{1-e^2\cos^2\theta} \text{ (Art. 70, Cor.).} \qquad (1)$$

SCH.—Equation (1) shows that, for every value of θ, r has two values, numerically equal, with contrary signs. These two values of r, taken together, make the diameter; hence, every diameter of the ellipse is bisected at the centre (see Art. 71, Sch.).

Also, it is evident from (1) that the value of r is the same for θ and $(\pi - \theta)$.

It is equally evident, from (1), that equal diameters make supplemental angles with the major axis.

Therefore, in the case of equi-conjugate diameters,

$$\theta' = \pi - \theta; \quad \text{and hence} \quad \tan \theta' = - \tan \theta;$$

which, in the equation of condition for conjugate diameters (Art. 81, Sch.), gives

$$\tan^2 \theta = \frac{b^2}{a^2} \qquad \text{or} \qquad \tan \theta = \pm \frac{b}{a}.$$

Hence, *the equi-conjugate diameters of an ellipse are the diagonals of the rectangle constructed on its two axes.*

Cor.—When $\theta = 0$ or $180°$, $r = \pm a$; when $\theta = 90°$ or $270°$, $r = \pm b$.

It is evident from equation (1) that r is the greatest possible when $\theta = 0$, giving $r = \pm a$; and the *least* possible when $\theta = 90°$, giving $r = \pm b$. Hence, in every ellipse the transverse axis is the *greatest*, and the conjugate axis is the *least* diameter. For this reason, the transverse and conjugate axes of an ellipse are called the *major* and *minor axes* respectively. (See Art. 71, Sch.)

98. *Any chord which passes through the focus of an ellipse is a third proportional to the major axis and the diameter parallel to the chord.*

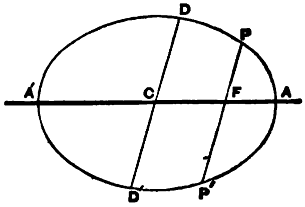

Fig. 79.

Let PP′ be any chord of the ellipse passing through the focus F; and DD′ the diameter parallel to PP′. Put $PF = r$, $P'F = r'$, and $AFP = \theta$.

Then (Art. 96),

$$r = \frac{a(1 - e^2)}{1 - e \cos \theta}, \quad \text{and} \quad r' = \frac{a(1 - e^2)}{1 + e \cos \theta}.$$

Therefore $$\text{PP}' = r + r' = \frac{2a(1-e^2)}{1-e^2\cos^2\theta}$$

$$= \frac{2b^2}{a(1-e^2\cos^2\theta)} \text{ (Art. 70, Cor.).} \quad (1)$$

From Art. 97, $$\overline{\text{CD}}^2 = \frac{b^2}{1-e^2\cos^2\theta}. \quad (2)$$

Dividing (1) by (2), we get

$$\frac{\text{PP}'}{\overline{\text{CD}}^2} = \frac{2}{a}, \quad \text{or} \quad \text{PP}' \cdot 2a = 4\overline{\text{CD}}^2;$$

therefore, $$2a : 2\text{CD} :: 2\text{CD} : \text{PP}'.$$

EXAMPLES.

1. Find the semi-axes of the ellipse $3y^2 + 2x^2 = 6$.

Comparing this equation with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we find

$$a = \sqrt{3}, \quad \text{and} \quad b = \sqrt{2}, \quad \textit{Ans.}$$

2. Find the semi-axes of the ellipse $4y^2 + 3x^2 = 19$.

Ans. $a = \sqrt{\tfrac{19}{3}}$, $b = \sqrt{\tfrac{19}{4}}$.

3. Find the points of intersection of the parabola $y^2 = 4x$ and the ellipse $3y^2 + 2x^2 = 14$. *Ans.* (1, 2) and (1, −2).

4. Find the equation of a tangent to the ellipse

$$3y^2 + 2x^2 = 35,$$

at the point whose abscissa is 2. *Ans.* $9y + 4x = 35$.

5. Find the eccentricity of the ellipse $2x^2 + 3y^2 = d^2$.

Ans. Eccentricity $= \sqrt{\tfrac{1}{3}}$.

6. Find the equation of the tangent to the ellipse at the end of the latus rectum; also, find the lengths of the intercepts of this tangent on the two axes.

Ans. $y + ex = a$; the intercepts are $\frac{a}{e}$ on the axis of x, and a on the axis of y.

7. Write the equation of the normal at the end of the latus rectum.

Ans. $y + ae^2 = \frac{x}{e}$.

8. Find the equation of the line through A′B, and also through CH (Fig. 78); and find the eccentricity of the ellipse if these two lines are parallel.

$$\textit{Ans.}\ \begin{cases} y = \frac{b}{a}(x + a); \quad y = \frac{b^2x}{a^2e}; \\ \text{the lines are parallel if } 2e^2 = 1. \end{cases}$$

9. Find a point on the ellipse such that the tangent at the point is equally inclined to the two axes.

Ans. $x = \frac{a^2}{\sqrt{a^2 + b^2}}, \quad y = \frac{b^2}{\sqrt{a^2 + b^2}}$.

10. Find a point on the ellipse such that the tangent at the point makes intercepts on the two axes that are proportional to the axes.

Ans. $x = \frac{a}{\sqrt{2}}, \quad y = \frac{b}{\sqrt{2}}$.

11. Express the equation of the tangent at any point of an ellipse in terms of the eccentric angle at that point.

Ans. $\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1$.

12. Find the angle (θ) at which the focal radius F′P (Fig. 78) is inclined to the major axis, when F′P is a mean proportional between the semi-axes of the ellipse, when $a = 50$ and $b = 30$.

Ans. $\cos\theta = \frac{5\sqrt{5} - 3\sqrt{3}}{4\sqrt{5}}$.

13. Show that the equations of the tangents to the ellipse $3x^2 + y^2 = 3$, and inclined at an angle of 45° to the major axis, are $y = x + 2$, $y = x - 2$.

14. If the semi-axes of an ellipse are 5 and 4, find the angle at which CP is inclined to the major axis, when it is an arithmetic mean between a and b.

Ans. $\cos\theta = \pm\frac{5}{27}\sqrt{17}$.

15. Find the length of the normal NP, in Fig. 69, and of RP. [See Art. 75, Cor.]

Ans. $NP = \frac{bb'}{a}$, and $RP = \frac{ab'}{b}$ (where a and b are the semi-axes, and b' is the semi-diameter conjugate to the diameter passing through the point P).

16. Prove that the equi-conjugate semi-diameter is to the semi-diagonal on the axes as 1 is to $\sqrt{2}$.

17. In the ellipse whose axes are 8 and 6, find the altitude of the circumscribed parallelogram whose sides are parallel to the equi-conjugate diameters.

[Find a' by Art. 86, Sch.; then alt. $=$ area $\div 2a'$, Cor.]
Ans. 6.79 nearly.

18. In an ellipse whose axes are 12 and 8, what is the length of the diameter from the point whose eccentric angle is 60°? *Ans.* $2\sqrt{21}$.

19. If from the vertex of any diameter right lines are drawn to the foci, prove that their product is equal to the square of half the conjugate diameter.

[This follows immediately from Arts. 72 and 84.]

20. Find the locus of the intersection of tangents at the extremities of conjugate diameters. *Ans.* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$.

[This is easily obtained by squaring and adding the equations of the two tangents, observing the relations of Art. 83. (See Salmon's Conic Sections, p. 198.)]

21. Find the locus of the intersection of a tangent with a perpendicular on it from either focus. *Ans.* $x^2 + y^2 = a^2$.

[This is readily obtained by writing the equation of the tangent in the form of (11) in Art. 74, and adding the square of it to the square of the equation of the perpendicular on it from either focus.]

22. Find the locus of the vertex of a triangle, having given the base $= 2m$, and the product of the tangents of the base angles $= \frac{l^2}{n^2}$.

[Take the base and a perpendicular to it at the middle point for the axes.] *Ans.* The locus is $l^2x^2 + n^2y^2 = l^2m^2$.

23. Find the polar of either focus of the ellipse; also, of either vertex of the minor axis. *Ans.* $x = \pm\frac{a}{e}$, $y = b$.

24. Find the equations of the tangent and normal at the extremity of the latus rectum; and determine the eccentricity of the ellipse in which the normal mentioned passes through the extremity of the minor axis.

$$\textit{Ans.}\begin{cases}\text{Equation of tangent is } ex + y = a\,; \\ \quad\text{“} \qquad \text{“ normal “ } x - ey = ae^3\,; \\ e = \sqrt{\dfrac{b}{a}}.\end{cases}$$

25. The ordinate of any point P on an ellipse is produced to meet the circumscribed circle at P′; prove that the focal perpendicular upon the tangent at P′ is equal to the focal distance of P.

[Combine Arts. 73 and 74 for the tangent at P′; then, Art. 24, $p =$ (after a little reduction) $a - ex = r$. (Art. 72.)]

26. In an ellipse, prove that the rectangle of the central perpendicular on any tangent, and the part of the corresponding normal intercepted between the axes, is constant, and equal to $a^2 - b^2$.

[By Art. 85, $p = \frac{ab}{b'}$; by Ex. 15, $\mathrm{NR} = \frac{b'(a^2 - b^2)}{ab}$; ∴ etc.]

27. Find the sum of the focal perpendiculars on the polar of (x', y').

[By Arts. 78 and 24,

$$p + p' = \frac{2a^2b^2}{\sqrt{a^4y'^2 + b^4x'^2}} = \frac{2ab}{\sqrt{\frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2}};$$

if (x', y') be *on* the ellipse, this value

$$= \frac{2ab}{\sqrt{a^2 - e^2x'}} = \frac{2ab}{b'} \text{ (Art. 84)};$$

if (x', y') be at the right focus, it equals

$$\frac{2ab}{be} = \frac{2a}{e}.$$

(Compare with Art. 71, Cor. 1.)]

28. Prove that the sum of the reciprocals of two focal chords at right angles to each other is constant.

[Find the focal chord PP′ by Art. 98, and the one perpendicular to it by putting sine for cosine; adding the reciprocals, we get $\frac{a(2 - e^2)}{2b^2}$ = a constant.]

29. If the axes of an ellipse be in the proportion of $\sqrt{2} : 1$, *any* parabola described on the minor axis as axis, and having its vertex at the centre, will cut the ellipse at right angles.

The equations of the ellipse and parabola are

$$a^2y^2 + b^2x^2 = a^2b^2, \tag{1}$$

and

$$x^2 = 2py, \tag{2}$$

respectively. Call θ and ϕ the angles which the tangents to the two curves at their point of intersection make with the axis of x; and φ' the angle which the tangent at (2) makes with the axis of y.

Then,

$$\tan\theta = -\frac{b^2x'}{a^2y'};$$

$$\tan\phi' = \frac{p}{x'} = \cot\phi;$$

hence, $$\tan\phi = \frac{x'}{p};$$

$$\therefore\quad \tan\theta\tan\phi = -\frac{b^2x'^2}{a^2py'}$$

$$= -\frac{2b^2}{a^2}\text{ (since } x'^2 = 2py'\text{)}. \quad (3)$$

Now, as $a : b :: \sqrt{2} : 1$, we have $a^2 = 2b^2$, which in (3) gives

$$\tan\theta\tan\phi = -1;$$

therefore the two tangents, and hence the two curves, at their point of intersection, cut each other at right angles. [See O'Brien's Co-ordinate Geometry, p. 128, where this example is incorrectly solved.]

30. Putting ρ and ρ' to denote the focal radii of any point on an ellipse, and ϕ for its eccentric angle, prove that

$$\rho = a(1 - e\cos\phi),$$
$$\rho' = a(1 + e\cos\phi).$$

31. From the centre of an ellipse, two radii-vectores are drawn at right angles to each other, and tangents to the curve are formed at their extremities; prove that the tangents intersect on the ellipse

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} + \frac{1}{b^2}.$$

CHAPTER VII.

THE HYPERBOLA.

99. The **Hyperbola** is the locus of a point moving in a plane so that its distance from a fixed point bears a constant ratio to its distance from a fixed right line, the ratio being greater than unity.*

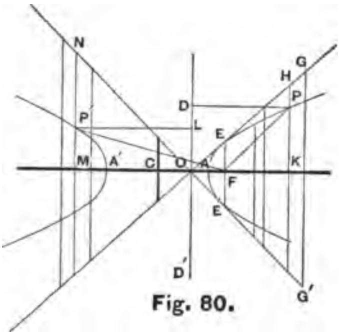

Fig. 80.

From this definition the hyperbola may be constructed by points, thus:

Let F be the fixed point, DD' the fixed right line, and e the given ratio. Draw through F the line OAF perpendicular to EE', parallel to DD'. Take

$$FE\ (= FE') : FO :: e : 1,$$

and draw OE and OE' produced indefinitely. Draw parallels to EE', meeting the lines OG and OG'. With the half of any one of these parallels, as KH, for a radius, and the fixed point F for a centre, describe an arc cutting KH at P; this is a point of the curve. For, joining P and F, and drawing PD perpendicular to DD', we have

$$KH\ (= FP) : KO\ (= PD) :: FE : FO;$$

that is, by construction we have

$$FP : PD :: e : 1.$$

In the same way, any required number of points in the curve may be found.

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* See Todhunter's Conic Sections, p. 188.

> 1, the distance from F to any point in the greater than the distance from the same point to DD′; therefore there are points in the curve on the te side of DD′, which are found in the same way as those to the right of DD′, thus: with the half of any of the parallels on the left of DD′, as MN, for a radius, and F for a centre, describe an arc cutting MN at P′; this is a point of the curve. For, joining P′ and F, and drawing P′L perpendicular to DD′, we have

$$\text{MN} \ (= \text{FP}') : \text{MO} \ (= \text{P}'\text{L}) \ :: \ \text{FE} : \text{FO};$$

that is, by construction we have $\text{FP}' : \text{P}'\text{L} \ :: \ e : 1.$

In the same way, any required number of points may be found. Connecting these points by a line, we have the required hyperbola.

The fixed line DD′ is called the **Directrix**; the fixed point F is called the **Focus**; OG and OG′ are called the **Focal Tangents**.

100. *To find the equation of the hyperbola.*

Let F be the focus, and YY′ the directrix; through F draw OX perpendicular to YY′; take OX and OY for the coordinate axes. Let (x, y) be any point P on the curve; join FP; draw PM and PD respectively perpendicular to OX and OY.

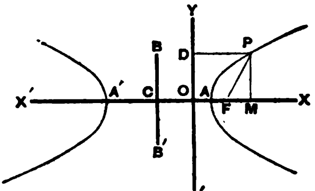

Fig. 81.

Represent OF by p, and the ratio of FP to PD by e. Then we have, from definition,

$$\text{FP} = e \cdot \text{PD}, \qquad \text{or} \qquad \overline{\text{FP}}^2 = e^2 \cdot \overline{\text{PD}}^2;$$

or

$$\overline{\text{FM}}^2 + \overline{\text{MP}}^2 = e^2 \cdot \overline{\text{PD}}^2;$$

that is,

$$(x - p)^2 + y^2 = e^2x^2, \tag{1}$$

which is the required equation.

COR.—When $y = 0$, we have $x - p = \pm ex$; therefore,

$$x = \frac{p}{1 \mp e},$$

which shows that the curve cuts the axis of x at the two points A and A$'$, giving

$$\text{OA} = \frac{p}{1+e}, \qquad \text{and} \qquad \text{A}'\text{O} = -x = \frac{p}{e-1}.$$

Since $e > 1$, $1 - e$ is a negative quantity; therefore OA$'$ must be measured to the *left* of the origin.

A and A$'$ are called the **Vertices** of the hyperbola, and C, the point midway between them, is called the **Centre** of the hyperbola.

$$\text{The distance AA}' = \text{A}'\text{O} + \text{OA}$$

$$= \frac{p}{e-1} + \frac{p}{1+e} = \frac{2ep}{e^2-1};$$

it is customary to denote this distance by $2a$. Therefore we have

$$2a = \frac{2ep}{e^2-1}.$$

101. Transform equation (1) of Art. 100 to A (Fig. 81).

The formulæ for this transformation become (Art. 100, Cor.),

$$x = \frac{p}{1+e} + x', \qquad y = y',$$

which in (1) gives

$$\left(\frac{p}{1+e} + x' - p\right)^2 + y'^2 = e^2\left(\frac{p}{1+e} + x'\right)^2,$$

or $$x'^2 + y'^2 - 2\frac{ep}{1+e}x' = e^2\left(\frac{2p}{1+e}x' + x'^2\right);$$

therefore, $$y'^2 = 2epx' - (1-e^2)x'^2$$

$$= (1-e^2)\left(\frac{2ep}{1-e^2}x' - x'^2\right);$$

therefore (Art. 100, Cor.),

$$y'^2 = (e^2 - 1)(2ax' + x'^2),$$

or, suppressing the accents, since the variables are general, we have

$$y^2 = (e^2 - 1)(2ax + x^2). \tag{1}$$

Cor.—When $x = -a$, (1) becomes

$$y^2 = -(e^2 - 1)a^2 = -b^2 \text{ [by putting } (e^2 - 1)a^2 = b^2].$$

Therefore $$e^2 - 1 = \frac{b^2}{a^2},$$

which in (1) becomes

$$y^2 = \frac{b^2}{a^2}(2ax + x^2). \tag{2}$$

102. Transform (2) of Art. 101, Cor., to C (Fig. 81).

The formulæ for this transformation become

$$x = x' - a, \qquad y = y',$$

which in (2) gives, after suppressing accents,

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2), \tag{1}$$

or $$a^2y^2 - b^2x^2 = -a^2b^2, \tag{2}$$

which may be more symmetrically written,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{3}$$

a form analogous to that of the equation of a right line (Art. 22, II).

Cor. 1.—By definition, $AF = e \cdot AO$, and $OA = \frac{p}{1 + e}$ (Art. 100, Cor.); therefore,

$$AF = \frac{ep}{1 + e} = \frac{e - 1}{e^2 - 1}ep = a(e - 1) \text{ [Art. 100, Cor.]};$$

that is, $$AF = a(e - 1). \tag{4}$$

$$OA = \frac{p}{1+e} = \frac{ep}{e^2-1}\cdot\frac{(e-1)}{e}$$

$$= \frac{a(e-1)}{e} \text{ [Art. 100, Cor.]};$$

that is, $$OA = \frac{a(e-1)}{e}. \tag{5}$$

$$CF = CA + AF = a + a(e-1) = ae;$$

that is, $$CF = ae. \tag{6}$$

$$CO = CA - AO = a - \frac{a(e-1)}{e} = \frac{a}{e};$$

that is, $$CO = \frac{a}{e}. \tag{7}$$

$$OF = p = CF - CO = ae - \frac{a}{e};$$

that is, $$OF = p = \frac{a(e^2-1)}{e}. \tag{8}$$

Cor. 2.—When $y = 0$, $x = \pm a$, which shows that the curve cuts the axis of x at two points equally distant from the origin, and on opposite sides of it. When $x = 0$, $y = \pm b\sqrt{-1}$; hence the curve cuts the axis of y in two *imaginary* points on opposite sides of the origin. We may, however, take two points B and B′, on different sides of C, making $CB = CB' = b$, as we shall have occasion to use them hereafter.

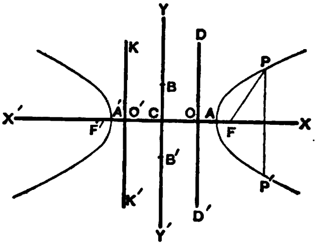

Fig. 82.

Cor. 3.—Solving (1) for y, we get

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2},$$

which shows that, for every value of $x > +a$ or $< -a$, there are two *real* values of y, numerically equal, with contrary signs; hence, for every point P on one side of the axis of x, there is a point P′ on the other side of the axis, at the same distance from it; and therefore the curve is symmetrical with respect to the axis of x. When x is $+a$ or $-a$, $y = \pm 0$; and for every value of x between $+a$ and $-a$, the two values of y are imaginary; therefore the curve is limited towards the centre by two tangents at A and A′.

Similarly, solving (1) for x, we get

$$x = \pm \frac{a}{b}\sqrt{y^2 + b^2},$$

which shows that for every value of y from $-\infty$ to $+\infty$ there are two real values of x, numerically equal with contrary signs; hence the curve is symmetrical with respect to the axis of y, and is unlimited in the direction of this axis.

SCH.—Because the curve is symmetrical with respect to the line BB′, it follows that if we take CF′ = CF (Fig. 82), and CO′ = CO, and draw KK′ perpendicular to OO′, the point F′ and the line KK′ will form respectively a second focus and directrix.

AA′ is called the **Transverse** axis of the hyperbola; BB′ is called the **Conjugate** axis of the hyperbola. In the *ellipse*, the conjugate axis is always less than the transverse axis (see Art. 70, Cor.), and therefore the former was called the *minor* and the latter the *major* axis. In the *hyperbola*, the conjugate axis may be greater than the transverse, since $b^2 = a^2(e^2 - 1)$ (Art. 101, Cor.), and e is > 1; therefore we do not call the axes in the hyperbola the *major* and *minor* axes.

The ratio e (Art. 99) is called the **Eccentricity** of the hyperbola.

The point C is called the **Centre** of the hyperbola, because it bisects every chord of the hyperbola which passes through it. This may be shown in the same way as in the case of the ellipse (Art. 71, Sch.).

COR. 4.—*To find the latus rectum* (Art. 53, Cor. 3).

Make $x = \mathrm{CF} = ae$ (Cor. 1); denote the corresponding value of y by p; we have from Eq. (1) (Art. 102),

$$p^2 = \frac{b^2}{a^2}(a^2e^2 - a^2) = b^2(e^2 - 1) = \frac{b^4}{a^2} \text{ (Art. 101, Cor.).}$$

Therefore, $$2p = \frac{2b^2}{a} = \frac{4b^2}{2a} = \textit{latus rectum.}$$

Forming a proportion from this equation, we have

$$2a : 2b :: 2b : 2p.$$

That is, *the latus rectum is a third proportional to the tranverse axis and the conjugate.*

Since $b^2 = (e^2 - 1)a^2$ (Art. 101, Cor.), we have

$$a^2 + b^2 = a^2e^2;$$

that is, $$a^2 + b^2 = \overline{\mathrm{CF}}^2 \text{ (Art. 102, Cor. 1).}$$

But $$a^2 + b^2 = \overline{\mathrm{AB}}^2 \text{ (see Fig. 82).}$$

Therefore, $$\mathrm{AB} = \mathrm{CF}.$$

Hence, *the conjugate axis of the hyperbola is a perpendicular to the transverse axis at its centre, and is limited by an arc described with the vertex of the transverse axis as a centre, and with a radius equal to the distance from the focus to the centre.*

COR. 5.—Comparing equation (2), Art. 102, with (2) of Art. 71, we see that the equation of the hyperbola may be derived from that of the ellipse, by changing $+b^2$ into $-b^2$. Hence, we infer that *any function of b, expressing a property of the ellipse, will be converted into one expressing a corresponding property of the hyperbola, by changing b into $b\sqrt{-1}$*; therefore, in obtaining the properties of the hyperbola that are similar to those which have been proved for the ellipse, we shall, in most cases, either change the sign of b^2, or else refer the student to the corresponding demonstration in the ellipse.

By a process similar to that of Art. 71, Cor. 5, the details of which the student must supply, we obtain

$$y'^2 : y''^2 :: (x' + a)(x' - a) : (x'' + a)(x'' - a);$$

that is, *the squares of any two ordinates to the transverse axis of an hyperbola are to each other as the rectangles of the segments into which they divide the transverse axis.*

Cor. 6.—A point is *outside, on,* or *inside* the hyperbola, according as $a^2y^2 - b^2x^2 + a^2b^2 >, =,$ or < 0. The proof is similar to that given in Art. 71, Cor. 6, for the ellipse.

A point is said to be *outside* the hyperbola if it lies in the space between the branches, so that no right line can be drawn through it to a focus without cutting the curve.

103. *To find the distance of any point in the hyperbola from the focus, in terms of the abscissa of the point.*

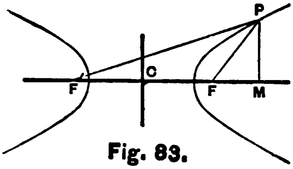

Fig. 83.

From the figure we have

$$\overline{FP}^2 = (x - ae)^2 + y^2$$

$$= (x - ae)^2 + \frac{b^2}{a^2}x^2 - b^2.$$

(Art. 102.)

$$= a^2 - 2aex + e^2x^2 \text{ (since } a^2e^2 - b^2 = a^2);$$

therefore, $\qquad FP = ex - a.$

[We take only the *positive* value of the root, for the reason given in Art. 72.]

In like manner we find, by writing $-ae$ for $+ae$,

$$\overline{F'P}^2 = (x + ae)^2 + y^2 = a^2 + 2aex + e^2x^2;$$

therefore, $\qquad F'P = ex + a.$

Hence, $\qquad F'P - FP = 2a;$

or, *the difference of the distances of any point in an hyperbola from the foci is equal to the transverse axis.*

COR.—This result furnishes two other methods of constructing an hyperbola, having given the axes.

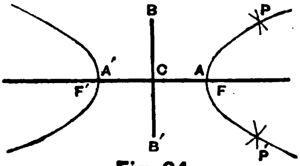

Fig. 84.

I. With C as a centre and BA as a radius, describe an arc cutting AA′ produced at F and F′; these points are the foci (Art. 102, Cor. 4). Now, with F′ as a centre and a radius greater than F′A, describe an arc; then with F as a centre, and a radius equal to that used before, diminished by the transverse axis AA′, describe another arc cutting the first at the point P; this will be a point of the curve, since

$$FP = F'P - 2a,$$

or

$$F'P - FP = 2a.$$

In the same way, any number of points may be found; joining these points, it will be a branch of the required hyperbola. By using F for the first centre and F′ for the second, with the same distances as before, any number of points of the other branch may be found.

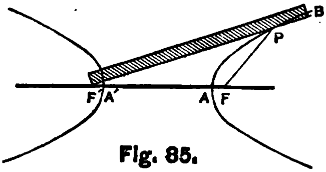

Fig. 85.

II. Take a ruler, and fasten one end of it at F′, so it can revolve about F′ as a centre. Take a string whose length is less than that of the ruler by AA′, and fasten one end of it at F and the other end at B, the end of the ruler; then press the string against the edge of the ruler with the point of a pencil P, and revolve the ruler about F′, keeping the string tight; the pencil will describe one branch of an hyperbola, since, in every position of it, we shall have

$$F'P - FP = AA'.$$

104. A **Conjugate Hyperbola** is one having the conjugate axis of a given hyperbola for its transverse axis, and the transverse axis of the given hyperbola for its conjugate axis. Either of two hyperbolas thus related is conjugate to the other. Thus, the hyperbola whose transverse axis is BB′ (Fig. 86) is the conjugate of the hyperbola whose transverse axis is AA′, and *conversely*, the latter is the conjugate of the former. They are often distinguished as the x **Hyperbola** and the y **Hyperbola**, each taking the name of the coordinate axis upon which its transverse axis lies; and when spoken of together are called **Conjugate Hyperbolas**.

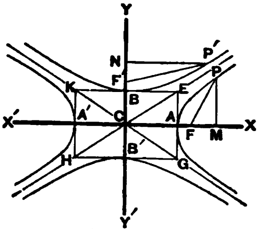

Fig. 86.

105. *To find the equation of an hyperbola conjugate to a given hyperbola.*

By Art. 102, the equation of the given hyperbola is

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2),$$

or, Fig. 86,
$$\overline{PM}^2 = \frac{\overline{CB}^2}{\overline{CA}^2}(\overline{CM}^2 - \overline{CA}^2).$$

Hence, since P′ is a point on the conjugate hyperbola, having BB′ for its transverse axis and AA′ for its conjugate axis, we have,

$$\overline{NP'}^2 = \frac{\overline{CA}^2}{\overline{CB}^2}(\overline{CN}^2 - \overline{CB}^2),$$

or
$$x^2 = \frac{a^2}{b^2}(y^2 - b^2), \tag{1}$$

which is the equation of the conjugate hyperbola, and is the same expression we would obtain from the equation of the given hyperbola by putting $-b^2$ for $+b^2$, and $-a^2$ for $+a^2$.

SCH. 1.—Since the second hyperbola holds the same relation to the axis of y that the first does to the axis of x, we might have deduced the equation of the y hyperbola at once by changing a to b and b to a, x to y and y to x in the equation of the x hyperbola.

The sides of the rectangle described on the axes are the tangents to the four branches at the vertices.

Comparing the equation of the conjugate hyperbola with that of the given hyperbola, we have (Art. 101, Cor.),

$$(e^2 - 1) b^2 = a^2,$$

or

$$a^2 + b^2 = b^2e^2 = \overline{\mathrm{CF}'}^2.$$

(See Art. 102, Cor. 1.) Therefore the foci of the y hyperbola are at the same distance from the centre as the foci of the x hyperbola, but the *eccentricity* of the former has a different value from that of the latter.

SCH. 2.—The equations of the diagonals CE and CG are respectively

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x.$$

If in the equations of the two conjugate hyperbolas we make $b = a$, we have (Art. 101, Cor.),

$$y^2 - x^2 = -a^2, \tag{2}$$

and (1) of the present Art. becomes

$$y^2 - x^2 = a^2. \tag{3}$$

These hyperbolas are called **Equilateral** hyperbolas, from the equality of the axes. The equilateral hyperbola corresponds to the case in which the ellipse becomes a circle. (See Art. 71, Cor. 7.) The peculiarity in the figure of the equilateral hyperbola is that *the curve is identical in form with its conjugate.* From Art. 101, Cor., we have

$$e^2 - 1 = \frac{b^2}{a^2};$$

therefore, in the *equilateral* hyperbola we have $e = \sqrt{2}$.

106. *To construct a pair of conjugate hyperbolas whose axes are given.*

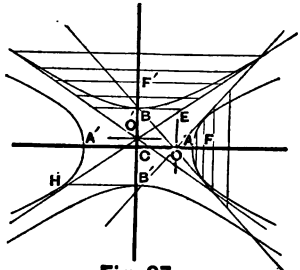

Fig. 87.

Draw the axes AA′ and BB′ at right angles to each other; construct the x hyperbola as in Art. 99. Now take CF′ = CF, which equals AB (Art. 105, Sch.), and F′ is the focus of the y hyperbola. Take BE = BF′, and B′H = B′F′; draw through E and H a right line; it is one of the focal tangents. Through O′ draw a line perpendicular to BB′; this is the directrix corresponding to the focus F′ of the y hyperbola. The construction is now the same as in Art. 99.

107. *To find the equation of the tangent at any point of an hyperbola.*

To obtain this equation for the hyperbola, we change b^2 into $-b^2$ in equations (6), (7), and (11) of Art. 74, and get

$$a^2yy' - b^2xx' = -a^2b^2, \tag{1}$$

$$y = \frac{b^2x'}{a^2y'}x - \frac{b^2}{y'}, \tag{2}$$

$$y = mx \pm \sqrt{a^2m^2 - b^2}. \tag{3}$$

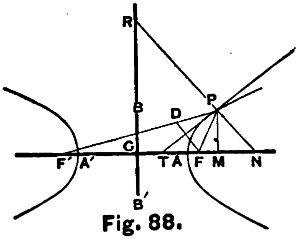

Fig. 88.

Cor.—To find the point in which the tangent cuts the axis of x, make $y = 0$ in (1), and get

$$x = \frac{a^2}{x'} = \text{CT},$$

which is the same value we found for the abscissa of the point at which the tangent cuts the axis of x in the ellipse.

(Art. 74, Cor. 2). This value of x has the same sign as x'; hence, for the right-hand branch, it is always positive; that is, the tangent to the right-hand branch cuts the axis of x to the right of the centre.

By Art. 102, Cor. 2, we have $F'C = FC = ae$; therefore we have

$$F'T = ae + \frac{a^2}{x'} = \frac{a}{x'}(ex' + a),$$

and $$FT = ae - \frac{a^2}{x'} = \frac{a}{x'}(ex' - a).$$

Hence, $F'T : FT :: ex' + a : ex' - a :: F'P : FP$

(by Art. 103). That is, *the tangent of an hyperbola cuts the distance between the foci in segments proportional to the adjacent focal radii of contact; and therefore it bisects the internal angle between these focal radii.*

This principle affords a method of drawing a tangent to an hyperbola at a given point.

Let P be the given point (see Fig. 88). Draw the focal radii F′P and FP to the given point P. On the longer, F′P, lay off PD = PF, and join DF. Through P draw PT perpendicular to DF; PT will be the trangent required, for it bisects the angle FPF′.

The *subtangent* $MT = CM - CT = x' - \frac{a^2}{x'}$. That is,

$$\textit{the subtangent} = \frac{x'^2 - a^2}{x'}.$$

108. *To find the equation of the normal at any point of an hyperbola.*

We change b^2 into $-b^2$ in (2) of Art. 75, and get

$$y - y' = -\frac{a^2y'}{b^2x'}(x - x'), \tag{1}$$

which is the required equation of the normal at (x', y').

COR. 1.—To find the point in which the normal cuts the axis of x, we make $y = 0$ in (1), and get, after reduction

$$x = \frac{a^2 + b^2}{a^2} x' = \text{CN (Fig. 88)} = e^2 x' \text{ (Art. 101, Cor.).}$$

The subnormal $\text{MN} = \text{CN} - \text{CM}$

$$= \frac{a^2 + b^2}{a^2} x' - x'^2 = \frac{b^2}{a^2} x'.$$

SCH.—The expression $\text{CN} = e^2 x'$ enables us, as in the case of the ellipse (Art. 75, Sch.), to draw a normal at any point P of the hyperbola, or one from any point N of the transverse axis.

COR. 2.—By Art. 102, Cor. 1,

$$\text{F}'\text{C} = \text{FC} = ae;$$

therefore we have

$$\text{F}'\text{N} = e\,(ex' + a),$$

and

$$\text{FN} = e\,(ex' - a).$$

Hence, $\text{F}'\text{N} : \text{FN} :: ex' + a : ex' - a :: \text{F}'\text{P} : \text{FP}$

(Art. 103). That is, *the normal of an hyperbola cuts the distance between the foci in segments proportional to the adjacent focal radii of contact; and hence it bisects the external angle between the focal radii of contact.*

109. *To find the locus of the intersection of the tangent at any point with the perpendicular on it from either focus.*

Changing the sign of b^2 in (3) and (4) of Art. 76, and adding the squares of the resulting equations together, we get

$$x^2 + y^2 = a^2,$$

for the required locus, which is therefore a circle described on the transverse axis.

110. *To find the co-ordinates of the point of contact of a tangent to an hyperbola from a fixed point.*

Let (x', y') be the required point of contact, and (x'', y'') the fixed point through which the tangent passes.

Changing $+ b^2$ to $- b^2$ in the results of Art. 77, we get

$$x' = \frac{a^2b^2x'' \mp a^2y''\sqrt{a^2y''^2 - b^2x''^2 + a^2b^2}}{b^2x''^2 - a^2y''^2},$$

$$y' = \frac{a^2b^2y'' \mp b^2x''\sqrt{a^2y''^2 - b^2x''^2 + a^2b^2}}{b^2x''^2 - a^2y''^2}.$$

These values indicate that from any fixed point *two* tangents can be drawn to an hyperbola, *real, coincident,* or *imaginary,* according as

$$a^2y''^2 - b^2x''^2 + a^2b^2 >, =, \text{ or } < 0;$$

that is, according as the point (x'', y'') is *outside, on,* or *inside* the curve (Art. 102, Cor. 6).

Cor.—It is clear that if any two real tangents be drawn from a given point to touch the *same branch,* their abscissas of contact will have *like* signs; and *unlike,* if they touch *different* branches. Hence, since the values of x in the former case must have the same signs, we have, regarding only their *numerical* values,

$$a^2b^2x'' > a^2y''\sqrt{a^2y''^2 - b^2x''^2 + a^2b^2};$$

or squaring, transposing, and reducing, we have

$$y'' < \frac{b}{a}x'. \qquad (1)$$

But (Art. 105, Sch. 2) $y = \frac{b}{a}x$ is the equation of the diagonal of the rectangle formed upon the axes of the hyperbola; therefore, the ordinate of the point from which two tangents can be drawn to the *same branch* of an hyperbola must be less than the corresponding ordinate of the diagonal; that is, the point itself must be somewhere between the diagonals (CE, CG) or (CH, CK) produced, and the adjacent branch of the curve (Fig. 86). These diagonals produced are called **Asymptotes** of the hyperbola, which we shall consider in Art. 113. Hence, generally, the two tangents which can be drawn to an hyperbola from any external point, will both touch the *same branch*, if the external point be between that branch and the adjacent portions of the asymptotes; but if the external point be so placed that we cannot pass from it to the curve without crossing an asymptote, the two tangents touch *different branches* of the curve.

111. *Tangents are drawn to an hyperbola from a given external point; to find the equation of the chord of contact* (Art. 77).

Change b^2 into $-b^2$ in (5) of Art. 78, and get

$$a^2yy' - b^2xx' = -a^2b^2, \qquad (1)$$

which is the equation of the chord of contact.

112. *Through any fixed point a chord is drawn to an hyperbola, and tangents to the hyperbola are drawn at the extremities of the chord; to find the equation of the locus of the intersection of the tangents, when the chord is turned about the fixed point.*

Change b^2 into $-b^2$ in (3) of Art. 79, and get

$$a^2yy' - b^2xx' = -a^2b^2, \qquad (1)$$

which is the equation required, and the locus is a right line.

Sch.—The line (1) is called the **Polar** of the point (x', y') with regard to the hyperbola $a^2y^2 - b^2x^2 = -a^2b^2$, and the point (x', y') is called the **Pole** of the line.

The statements in Art. 49 with respect to the circle may all be applied to the hyperbola as they were to the parabola (Art. 61), and the same conclusions arrived at that were reached in Arts. 49 and 61, and referred to in the ellipse (Art. 79, Sch.).

113. An **Asymptote** of a curve is a line which continually approaches the curve, and becomes tangent to it only at an infinite distance, while it passes within a finite distance of the origin. We have called the diagonals produced of the rectangle on the axes (Art. 110, Cor.), the *asymptotes* of the hyperbola; we now proceed to show that they are such, that is, that they meet the curve only at infinity.

114. *To prove that the diagonals of the rectangle on the axes are asymptotes to both the given and conjugate hyperbolas.*

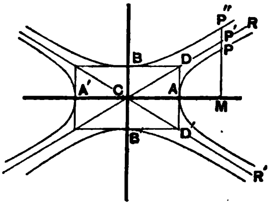

Fig. 89.

Produce the ordinate MP of any point P in the given hyperbola, to meet the diagonal CR and the conjugate hyperbola, in the points P′ and P″ respectively. The distance of the point P from CR = PP′ sin PP′C, and therefore it varies as PP′. Now, if CM, the common abscissa $= x$, $\text{PM} = y$, $\text{P'M} = y'$, and $\text{P''M} = y''$, we have, from the equations of the given hyperbola, the diagonal, and the conjugate hyperbola,

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2), \tag{1}$$

$$y'^2 = \frac{b^2}{a^2}x^2, \tag{2}$$

$$y''^2 = \frac{b^2}{a^2}(x^2 + a^2). \tag{3}$$

Subtracting (1) from (2), we have

$$y'^2 - y^2 = b^2, \qquad \text{or} \qquad y' - y = \frac{b^2}{y + y'}. \tag{4}$$

Subtracting (2) from (3), we have

$$y''^2 - y'^2 = b^2, \qquad \text{or} \qquad y'' - y' = \frac{b^2}{y'' + y'}. \tag{5}$$

If now we suppose the abscissa CM to increase continually, and the line MP to move parallel to itself, the ordinates y, y', and y'' will increase continually, and therefore, from (4) and (5), $y'-y$ and $y''-y'$ will diminish continually; and when x (CM), and therefore y, y', and y'' become infinitely great, $y'-y$ and $y''-y'$ will become infinitely small; that is, as x increases indefinitely, the two curves continually approach the diagonal CR, and become tangent to it and to each other only at infinity. Hence the diagonals are asymptotes to both curves.

COR. 1.—The equations of CR and CR′ are (Art. 105, Sch. 2),

$$y=\frac{b}{a}x \qquad \text{or} \qquad \frac{x}{a}-\frac{y}{b}=0;$$

and

$$y=-\frac{b}{a}x \qquad \text{or} \qquad \frac{x}{a}+\frac{y}{b}=0;$$

therefore the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$ includes both asymptotes.

COR. 2.—Let $ACR=\theta$, $ACR'=\theta'$; then

$$\tan\theta=\frac{b}{a}, \qquad \tan\theta'=-\frac{b}{a};$$

$$\sin\theta=\frac{b}{\sqrt{a^2+b^2}}, \qquad \cos\theta=\frac{a}{\sqrt{a^2+b^2}}=\frac{1}{e};$$

$$\sin\theta'=-\frac{b}{\sqrt{a^2+b^2}}, \qquad \cos\theta'=\frac{a}{\sqrt{a^2+b^2}}=\frac{1}{e}.$$

115. *To find the equation of any diameter.* (Def. of Art. 62.)

Change b^2 into $-b^2$ in (2) of Art. 80, and get

$$y=\frac{b^2}{a^2}\cot\theta\cdot x \qquad (1)$$

for the required equation.

Since a^2 and b^2 are constant for any given hyperbola, and θ is constant for any given system of parallel chords, (1) is the equation of a right line passing through the origin, that is, through the centre of the hyperbola. Hence, every diameter of the hyperbola passes through the centre. By giving to θ a suitable value, (1) may be made to represent *any* right line passing through the centre. Hence, *every* right line that passes through the centre of an hyperbola is a diameter; that is, it bisects some system of parallel chords.

SCH.—To draw a diameter of an hyperbola, draw any two parallel chords, and bisect them; the line passing through the points of bisection is a diameter. The intersection of two diameters will be the centre of the hyperbola.

COR. 1.—Let $\theta' =$ the inclination of the diameter itself to the transverse axis; then we have

$$\tan\theta' = \frac{y}{x};$$

which in (1) gives

$$\tan\theta\tan\theta' = \frac{b^2}{a^2},$$

as the relation between θ and θ' when they are the angles which a system of parallel chords and their diameter respectively make with the axis of x.

COR. 2.—Writing the equation of the diameter in the form

$$y = \tan\theta\cdot x, \qquad (1)$$

and eliminating y between this equation and that of the given hyperbola, to find the abscissas of the points of intersection of (1) and the curve, we obtain

$$x = \pm\frac{ab}{\sqrt{b^2 - a^2\tan^2\theta}}. \qquad (2)$$

Now, eliminating y between (1) and the equation of the conjugate hyperbola (Art. 105), to find the abscissas of the points of intersection of (1) and the conjugate curve, we obtain

$$x = \pm \frac{ab}{\sqrt{a^2 \tan^2 \theta - b^2}}. \tag{3}$$

If $a^2 \tan^2 \theta < b^2$, that is, if $\tan \theta < \pm \frac{b}{a}$, the values of x in (2) are *real*, showing that (1) intersects the given hyperbola at finite distances from the centre; while the values of x in (3) are *imaginary*, showing that (1) does not cut the y hyperbola.

If $a^2 \tan^2 \theta > b^2$, that is, if $\tan \theta > \pm \frac{b}{a}$, the values of x in (2) are *imaginary*, showing that (1) does not cut the given hyperbola; while the values of x in (3) are *real*, showing that (1) cuts the y hyperbola at finite distances from the centre.

If $a^2 \tan^2 \theta = b^2$, that is, if $\tan \theta = \pm \frac{b}{a}$, the values of x in (2) and (3) are infinite, showing that (1) does not cut either the x or the y hyperbola. In this case, (1) coincides with the diagonals of the rectangle described on the axes of the two conjugate hyperbolas (Art. 105), that is, with the asymptotes (Art. 113).

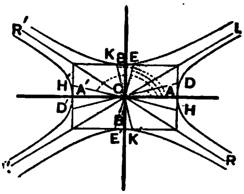

Fig. 90.

We learn, then, that diameters which cut the given hyperbola in real points, must either make with the transverse axis an angle *less* than is made by the first of these diagonals, or *greater* than is made by the second, as DD′ and HH′. If they cut the *conjugate* hyperbola in real points, they must either make with the transverse axis an angle *greater* than is made by the first of these diagonals, or *less* than is made by the second, as EE′ and KK′. If they *coincide* with these diagonals, as LL′ and RR′, they will intersect the hyperbolas at an infinite distance. Hence, every right line drawn through the centre of an hyperbola must meet the hyperbola or its conjugate, unless it coincides with one of the asymptotes.

116. *If one diameter of an hyperbola bisects all chords parallel to a second diameter, the second will bisect all chords parallel to the first.*

Let θ and θ' be the respective inclinations of any two diameters to the transverse axis. Then the condition that the first diameter shall bisect all chords parallel to the second diameter (Art. 115, Cor. 1) is

$$\tan\theta \tan\theta' = \frac{b^2}{a^2}. \tag{1}$$

But this is also the condition that the second diameter bisects all chords parallel to the first.

SCH.—Two diameters are **Conjugate** when each bisects all chords parallel to the other.

Because the conjugate of any diameter is parallel to the chords which that diameter bisects, therefore the inclinations of two conjugates must be connected in the same way as those of a diameter and its bisected chords. Hence, if θ and θ' are the inclinations, the *equation of condition for conjugate diameters* in the hyperbola (Art. 115, Cor. 1) is

$$\tan\theta \tan\theta' = \frac{b^2}{a^2}. \tag{2}$$

This condition shows that the tangents of inclination of any two conjugate diameters have *like* signs; therefore it indicates that the angles made with the transverse axis by the two conjugates are either *both acute* or *both obtuse.* Therefore, *conjugate diameters of an hyperbola lie on the same side of the conjugate axis,* as CD and CE, or CK and CH (see Fig. 90).

COR.—From (2), if

$$\tan\theta < \frac{b}{a}, \qquad \tan\theta' > \frac{b}{a};$$

and if $$\tan\theta > -\frac{b}{a} \qquad \tan\theta' < -\frac{b}{a}.$$

Therefore (Art. 115, Cor. 2), *if one of two conjugates,* DD′, *meets an hyperbola, the other,* EE′, *meets the conjugate hyperbola.*

117. ***The tangent at either extremity of any diameter is parallel to its conjugate diameter.***

[For demonstration, see Art. 82.]

118. *Given the co-ordinates x', y' of one extremity of a diameter, to find the co-ordinates x'', y'' of either extremity of the conjugate diameter.*

By the **Extremities** of the conjugate diameter, we mean the points in which the conjugate cuts the conjugate hyperbola.

Let (x', y') be the point D (Fig. 90), and (x'', y'') the point E or E′. Since the conjugate diameter EE′ is parallel to the tangent at (x', y') (Art. 117), and passes through the origin, therefore its equation (Art. 107) is

$$y = \frac{b^2x'}{a^2y'}x;$$

which, combined with the equation of the conjugate hyperbola (Art. 105) gives

$$x'' = \pm\frac{a}{b}y'; \qquad y'' = \pm\frac{b}{a}x'.$$

We see that the upper signs of the co-ordinates are both positive and the lower signs both negative, while in the ellipse (Art. 83), the upper signs are unlike and the lower also. This agrees with the properties of the two curves developed in Arts. 81 and 116, Sch.

119. *To express the length of a semi-diameter* (a'), *and its conjugate* (b'), *in terms of the abscissa of the extremity of the diameter.*

Let (x', y') and (x'', y'') be the extremities D and E of the diameters DD′ and EE′; then we have

$$a'^2 = x'^2 + y'^2 = x'^2 + \frac{b^2}{a^2}(x'^2 - a^2) \text{ (Art. 102)}$$

$$= \frac{a^2 + b^2}{a^2}x'^2 - b^2;$$

therefore $$a'^2 = e^2x'^2 - b^2 \text{ (Art. 101, Cor.).} \qquad (1)$$

Also, $$b'^2 = x''^2 + y''^2 = \frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2 \text{ (Art. 118)}$$

$$= x'^2 - a^2 + \frac{b^2}{a^2}x'^2 \text{ (Art. 102);}$$

therefore, $$b'^2 = e^2x'^2 - a^2 \text{ (Art. 101, Cor.).} \qquad (2)$$

Cor. (1) — (2) gives

$$a'^2 - b'^2 = a^2 - b^2;$$

that is, the difference of the squares of any two conjugate diameters of an hyperbola is equal to the difference of the squares of the axes.

120. *To find the length of the perpendicular from the centre to the tangent at any point.*

Let (x', y') be the point, and p the perpendicular. The equation of the tangent at (x', y') is (Art. 107),

$$a^2yy' - b^2xx' = -a^2b^2. \qquad (1)$$

Therefore (Art. 24),

$$p = \frac{a^2b^2}{\sqrt{a^4y'^2 + b^4x'^2}} = \frac{ab}{\sqrt{\frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2}} = \frac{ab}{b'}. \text{ (Art. 119.)}$$

121. *To find the angle between any pair of conjugate diameters.*

Let ϕ be the required angle ECD in Fig. 90. By the same process as in the ellipse (Art. 86), we find

$$\sin\phi = \frac{ab}{a'b'}. \tag{1}$$

Cor.—Clearing (1) of fractions, we have

$$a'b' \sin\phi = ab, \tag{2}$$

which shows that *the area of the parallelogram whose sides touch the hyperbola at the ends of any pair of conjugate diameters is constant, and equal to the rectangle of the axes.*

Sch.—By Art. 119, Cor., $a'^2 - b'^2 =$ a constant; therefore, a' and b' increase or decrease together; hence, by causing D to move along the hyperbola from A, E also will move along from B (Fig. 90). But any diameter CD tends towards an *infinite length*, as its inclination tends towards the limit $\theta = \tan^{-1}\frac{b}{a}$ (Art. 115, Cor. 2); therefore its semi-conjugate CE tends towards infinity; and, as $a'b' \sin\phi$ is *constant*, and a' and b' tend towards infinity, $\sin\phi$ tends towards 0; or, the angle between two conjugates of an hyperbola diminishes without limit. When the two conjugates approach infinity in length, they tend to coincide with the diagonals of the rectangle constructed on the axes; but they are never equal, since $a'^2 - b'^2$ is always equal to $a^2 - b^2$ (Art. 119, Cor.), unless the curve is equilateral. Therefore, the infinite diameters which form the limit of the conjugates are not *equal* infinites, and hence we do not, as in the ellipse, have *equi-conjugates.* We may, however, call these conjugates in their limit, when they coincide with each other and with either of the asymptotes, **Self Conjugates,**[*] since each is a diameter conjugate to itself.

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[*] See Howison's Analytic Geometry, p. 331.

The inclinations of the self-conjugate diameters to the transverse axis are determined by the equation

$$\tan\theta = \pm\frac{b}{a}. \quad \text{(Art. 115, Cor. 2.)}$$

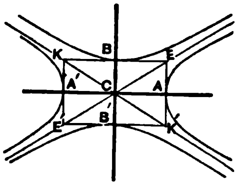

Fig. 91.

The first value corresponds to the angle ACE, and the second value to the angle ACK (Fig. 91).

The inclination of these self-conjugates to each other, as ECK or ECK′, is determined by

$$\sin\phi = 2\sin \text{BCE} \cos \text{BCE}$$
$$= 2\frac{a}{\sqrt{a^2+b^2}} \times \frac{b}{\sqrt{a^2+b^2}};$$

that is, $$\sin\phi = \frac{2ab}{a^2+b^2},$$

where $$\phi = \text{ECK or ECK}'.$$

122. *If a chord and diameter of an hyperbola are parallel, the supplemental chord and the conjugate diameter are parallel.* (See Def., Art. 88.)

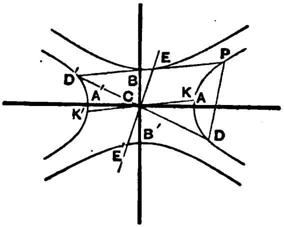

Fig. 92.

Let DD′ be a diameter of the hyperbola; PD and PD′ two supplemental chords, the first parallel to the diameter EE′; then will the supplemental chord PD′ be parallel to the conjugate diameter KK′.

Let (x', y') be the point D, and therefore $(-x', -y')$ will be the point D′. Let ϕ and ϕ' be the inclinations of the two chords DP and D′P. Then, by the same process as in Art. 89, or simply by changing b^2 into $-b^2$ in that Art., we get

$$\tan\phi\tan\phi' = \frac{b^2}{a^2},$$

as the condition that the two chords DP and D′P shall be supplemental.

Now, from Art. 116, Sch., we have

$$\tan\theta\tan\theta' = \frac{b^2}{a^2},$$

as the condition that two diameters shall be conjugate to each other; the rest of the argument of Art. 89 applies directly to the hyperbola. Therefore, the supplemental chord PD′ is parallel to the conjugate diameter KK′.

123. *To find the equation of the hyperbola referred to any pair of conjugate diameters.*

To do this we must transform the equation of the hyperbola

$$a^2y^2 - b^2x^2 = -a^2b^2, \qquad (1)$$

from rectangular to oblique axes, having the same origin.

Let DD′ and SS′ be two conjugate diameters. Take CD for the new axis of x, and CS for the new axis of y. Denote the angle ACD by θ and ACS by θ'. Let x, y be the co-ordinates of any point P of the hyperbola referred to the old axes, and x', y' the co-ordinates of the same point referred to the new axes.

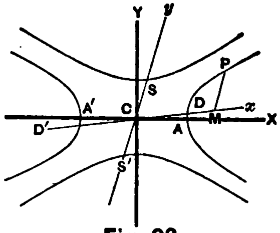

Fig. 93.

Now we may use the same process employed in Art 90; or, we may simply change b^2 into $-b^2$ in (3) of Art. 90 (see Art. 102, Cor. 5), and get

$$(a^2\sin^2\theta - b^2\cos^2\theta)\,x'^2 + (a^2\sin^2\theta' - b^2\cos^2\theta')\,y'^2 = -a^2b^2. \quad (1)$$

Let a' and b' denote the lengths of the semi-diameters CD and CS. If we make $y = 0$ in (1), we get

$$x'^2 = \frac{-a^2b^2}{a^2 \sin^2 \theta - b^2 \cos^2 \theta} = a'^2. \tag{2}$$

Also, if in (1) we make $x = 0$, we get

$$y'^2 = \frac{-a^2b^2}{a^2 \sin^2 \theta' - b^2 \cos^2 \theta'} = -b'^2. \tag{3}$$

We put this latter equal to $-b^2$, because we have supposed the new axis of x to meet the given hyperbola, as in Fig. 93; therefore we know (Art. 116, Cor.) that the new axis of y will *not* meet the given hyperbola; hence $\frac{-a^2b^2}{a^2 \sin^2 \theta' - b^2 \cos^2 \theta'}$ is a negative quantity.

From (2) we get $\quad a^2 \sin^2 \theta - b^2 \cos^2 \theta = -\frac{a^2b^2}{a'^2}. \qquad (4)$

From (3) we get $\quad a^2 \sin^2 \theta' - b^2 \cos^2 \theta' = \frac{a^2b^2}{b'^2}. \qquad (5)$

Substitute (4) and (5) in (1), divide by $-a^2b^2$, omit accents from the variables, and we get

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1, \tag{6}$$

or,

$$a'^2 y^2 - b'^2 x^2 = -a'^2 b'^2, \tag{7}$$

which is the equation required, and is of the same form as when referred to the axes of the curve (Art. 102).

Similarly, the equation of the *conjugate* hyperbola referred to the same pair of conjugate diameters is

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1, \tag{8}$$

or

$$a'^2 y^2 - b'^2 x^2 = a^2b^2. \tag{9}$$

[Let the student give the demonstration.]

Cor.—Comparing (7) with (9) of Art. 90, we see that the equation of the hyperbola may be derived from that of the ellipse by changing b^2 into $-b^2$. Hence, we infer that *any function of b' expressing a property of the ellipse will be converted into one expressing a corresponding property of the hyperbola by changing b' into $b'\sqrt{-1}$.*

124. *To find the equation of a tangent to the hyperbola referred to any pair of conjugate diameters.*

By reasoning exactly as in Art. 91, using the term "hyperbola" for "ellipse," or, by changing b'^2 into $-b'^2$ in (1) of Art. 91, according to Art. 120, Cor., we get

$$a'^2yy' - b'^2xx' = -a'^2b'^2, \qquad (1)$$

which is the required equation.

Cor.—To find where the tangent cuts the axis of x, make $y = 0$ in (1), and get

$$x = \frac{a'^2}{x'}.$$

125. *To prove that tangents at the extremities of any chord of an hyperbola meet on the diameter which bisects that chord.*

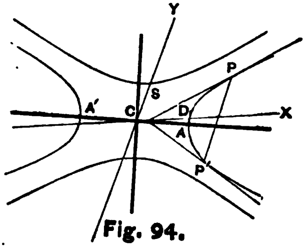

Fig. 94.

Take the diameter CD, which bisects the chord PP', for the axis of x, and the conjugate diameter CS for the axis of y.

Now reason as in Art. 92, or change b'^2 into $-b'^2$ in (1) and (2) of Art. 92, according to Art. 123, Cor., and get

$$a'^2yy' - b'^2xx' = -a'^2b'^2, \qquad (1)$$

and

$$-a'^2yy' - b'^2xx' = -a'^2b'^2, \qquad (2)$$

which are the equations of the tangents at the extremities of the chord PP′ referred to the diameter CD which bisects PP′, and the conjugate diameter CS. Now, by Art. 124, Cor., both of these tangents cut the axis of x at the point $\left(\frac{a'^2}{x'},\ 0\right)$, which proves the proposition.

126. *If tangents are drawn at the extremities of any focal chord of an hyperbola:*

I. *The tangents will intersect on the corresponding directrix.*

II. *The line drawn from the point of intersection of the tangents to the focus will be perpendicular to the focal chord.*

I. Reasoning as in Art. 93, we find for the equation of the chord of contact (Art. 111),

$$a^2yy' - b^2xx' = -a^2b^2, \tag{1}$$

which, for the right-hand focus $(ae, 0)$, becomes

$$-b^2aex' = -a^2b^2;$$

or

$$x' = \frac{a}{e}; \tag{2}$$

that is, the point of intersection of the tangents is on the corresponding directrix (Art. 102, Cor.), showing that the directrix is the polar of the focus. (Art. 79, Sch.)

II. The equation of the line passing through the right-hand focus and the point (x', y') is, by (Art. 26),

$$y = \frac{y'}{x' - ae}(x - ae). \tag{3}$$

From (2), $x' = \frac{a}{e}$, which in (3) gives

$$y = \frac{y'e}{a - ae^2}(x - ae)$$

$$= -\frac{aey'}{b^2}(x - ae). \text{ (Art. 101, Cor.)} \tag{4}$$

The equation of the chord of contact [see (1) above] is

$$y = \frac{b^2x'}{a^2y'}x - \frac{b^2}{y'};$$

which becomes, for the focal chord [since $x' = \frac{a}{e}$, from (2)],

$$y = \frac{b^2}{aey'}x - \frac{b^2}{y'}, \tag{5}$$

which is perpendicular to (4), by Art. 27, Cor. 1.

127. *Find the locus of the point of intersection of two tangents to an hyperbola at right angles to each other.*

Reason as in Art. 94, or change b^2 into $-b^2$ in equation (5) of that Art., and get

$$x^2 + y^2 = a^2 - b^2 \tag{1}$$

as the required locus. Hence, the locus is a circle with its centre at C, and with $\sqrt{a^2 - b^2}$ for its radius, *unless* $b^2 > a^2$, in which case the locus is impossible; that is, two tangents cannot be drawn at right angles to each other when b^2 is greater than a^2.

128. *The rectangle of the focal perpendiculars upon any tangent is constant, and equal to the square of the semi-conjugate axis.*

Call p and p' the perpendiculars. The equation of the tangent at any point (x', y') is

$$a^2yy' - b^2xx' = -a^2b^2.$$

By Art. 24,

$$p = +\frac{b^2x'ae - a^2b^2}{\sqrt{a^4y'^2 + b^4x'^2}} = \frac{b(ex' - a)}{\sqrt{\frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2}}$$

$$= \frac{b}{b'}(ex' - a). \quad \text{(Art. 119.)}$$

Also, $$p' = -\frac{-b^2x'ae - a^2b^2}{\sqrt{a^4y'^2 + b^4x'^2}} = \frac{b\,(ex' + a)}{\sqrt{\frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2}};$$

$$= \frac{b}{b'}(ex' + a). \quad \text{(Art. 119.)}$$

Hence, $$pp' = \frac{b^2}{b'^2}(e^2x'^2 - a^2) = b^2. \quad \text{(Art. 119.)}$$

129. *To find the polar equation of the hyperbola, the left focus being the pole.*

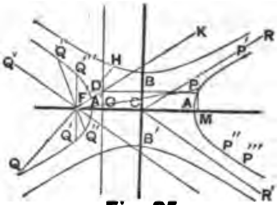

Fig. 95.

Let F'P $= r$; AF'P $= \theta$; then (Def. of Art. 99) we have,

$$\begin{aligned} \text{F'P} &= e \cdot \text{PD} \\ &= e\,(\text{F'M} - \text{F'O}) \\ &= e \cdot \text{MF'} - e \cdot \text{F'O} \\ &= e \cdot \text{F'P} \cos \text{AF'P} - a(e^2 - 1) \quad \text{(Art. 102, Cor. 1)}, \end{aligned}$$

or $$r = er\cos\theta - a\,(e^2 - 1);$$

therefore, $$r = \frac{a\,(e^2 - 1)}{e\cos\theta - 1}, \tag{1}$$

which is the equation required.

Cor.—When $\theta = 0$, $r = ae + a =$ F'C + CA = F'A, as it should do. (Art. 102, Cor. 1)

When $e\cos\theta - 1 = 0$, that is, when $\theta = \cos^{-1}\frac{1}{e}$,

$$r = \frac{a\,(e^2 - 1)}{0} = \infty.$$

But in this case r, or F'K, is parallel to the asymptote CR (See Art. 114, Cor, 2, and Fig. 89). That is, while θ increases from 0 to $\cos^{-1}\frac{1}{e}$, r increases from $ae + a$ to ∞, tracing the branch APP'.

When θ passes the value $\cos^{-1}\frac{1}{e}$, $e\cos\theta - 1$ becomes negative, and therefore r becomes negative, and the left-hand branch is generated, the negative end of r tracing QQ′Q″; thus, when $\theta = \text{AF}'\text{H}$, r, being negative, is reckoned backwards to Q.

When $\theta = 90°$, $r = -a(e^2-1) = -\frac{b^2}{a}$ (Art. 101, Cor.), which equals the *semi latus rectum*, p, with a negative sign (Art. 102, Cor. 4), and Q′ is located.

When $\theta = 180°$, $r = -a(e-1) = a - ae = -\text{F}'\text{A}'$, as it should.

While θ increases from 90° to 270°, the arc Q′Q″A′Q‴ is traced with the negative end of r.

When $\theta = 270°$, $r = -a(e^2-1) = -\frac{b^2}{a^2} = -p$, and the point Q‴ is *located.*

While θ increases from 270° to $\cos^{-1}\frac{1}{e}$, r remains negative, and increases numerically from p to ∞, its negative end tracing Q‴, Q$^{\text{iv}}$.

At $\theta = \cos^{-1}\frac{1}{e}$ in the fourth quadrant, $r = \infty$, and is parallel to the asymptote CR′.

While θ increases from $\cos^{-1}\frac{1}{e}$ to 360°, r is positive, and diminishes from ∞ to $a + ae$, and the arc P‴, P″, A is traced.

130. *To find the polar equation of the hyperbola when the pole is at the centre.*

Changing $a^2y^2 - b^2x^2 = -a^2b^2$ into a system of polar co-ordinates (as in Art. 97), we have

$$r^2 = \frac{a^2b^2}{b^2\cos^2\theta - a^2\sin^2\theta} = \frac{b^2}{e^2\cos^2\theta - 1} \text{ (Art. 101, Cor.) } (1)$$

Similarly, the polar equation of the conjugate hyperbola is

$$r^2 = \frac{b^2}{1 - e^2 \cos^2 \theta}. \qquad (2)$$

Cor.—Equation (1) shows that for every value of θ between $-\cos^{-1}\frac{1}{e}$ and $+\cos^{-1}\frac{1}{e}$, r has two *real* values, numerically equal, with contrary signs. These two values of r, taken together, make the diameter; hence, every diameter of the hyperbola is bisected at the centre (Art. 102, Sch.).

When $\theta = \cos^{-1}\frac{1}{e}$, $r = \infty$; but in this case, r or CR, Fig. 89, coincides with the asymptote.

While θ increases from $\cos^{-1}\frac{1}{e}$ to $\left(360° - \cos^{-1}\frac{1}{e}\right)$, r is imaginary, showing that it does not reach either branch of the given hyperbola.

Equation (2) shows that for every value of θ between $\cos^{-1}\frac{1}{e}$ and $\left(360° - \cos^{-1}\frac{1}{e}\right)$, r has two real values, numerically equal, with contrary signs. These two values of r, taken together, make the diameter of the conjugate hyperbola; hence, every diameter of the conjugate hyperbola is bisected at the centre.

When $\theta = \cos^{-1}\frac{1}{e}$, $r = \infty$; in this case, r coincides with the asymptote.

For every value of θ between $-\cos^{-1}\frac{1}{e}$ and $+\cos^{-1}\frac{1}{e}$, r is imaginary, showing that it does not reach either branch of the given hyperbola.

In (1), r is least when $\theta = 0$, giving

$$r = \sqrt{\frac{b^2}{e^2 - 1}},$$

which equals a (Art. 101, Cor.). In (2), r is least when $\theta = 90°$, giving $r = b$. Hence, in the hyperbola, each axis is the minimum diameter of its own curve.

Also, it is evident from both (1) and (2) that the value of r is the same for θ and $(\pi - \theta)$. Therefore, diameters which make supplemental angles with the transverse axis of an hyperbola are equal.

131. The properties of the hyperbola hitherto established are similar to those of the ellipse. We have now to consider some properties *peculiar to the hyperbola*, arising from the presence of the *asymptotes*. (See Art. 113.)

132. *To prove that the asymptotes are the diagonals of every parallelogram formed on a pair of conjugate diameters.*

The equations of the hyperbola and its asymptotes, when referred to the axes of the curve, are respectively

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{1}$$

and

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0. \tag{2}$$

When we transform the equation of the hyperbola to its conjugate diameters (Art. 123), equation (1) becomes

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1;$$

therefore we may at once infer that (2) transformed to the same conjugate diameters, becomes

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 0;$$

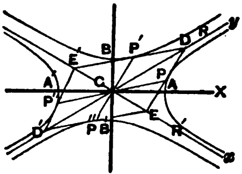

Fig. 96.

that is, the equations of the asymptotes CR and CR′, referred to any pair of conjugate diameters, are

$$\frac{x}{a'} - \frac{y}{b'} = 0, \tag{3}$$

and
$$\frac{x}{a'} + \frac{y}{b'} = 0. \tag{4}$$

Take CP $= a'$, and CP$' = b'$.

Equation (3), or $y = \frac{b'}{a'}x$, is the equation of a line passing through the origin and the point (a', b') (see Art. 26, Cor. 4), that is, through C and D; and (4), or $y = -\frac{b'}{a'}x$, is a line passing through the origin and $(a', -b')$, that is, through C and E. Hence, (3) and (4), which are the asymptotes, are also the diagonals of the parallelogram EDE′D′ on the conjugate diameters PP″ and P′P‴.

133. *To find the equation of the hyperbola referred to its asymptotes as axes.*

To do this, we must transform the equation

$$a^2y^2 - b^2x^2 = -a^2b^2 \tag{1}$$

from rectangular to oblique axes, having the same origin.

Let CX and CY be the old axes (Fig. 96). Take the lower asymptote CR′ for the new axis of x, and the other, CR, for the new axis of y.

Let x, y be the co-ordinates of any point P in the curve referred to the old axes, and x', y' the co-ordinates of the same point referred to the new axes. Denote the angles ACR′ and ACR by θ and θ' respectively.

The formulæ for transformation (Art. 35, Cor. 1) are

$$x = x' \cos\theta + y' \cos\theta',$$
$$y = x' \sin\theta + y' \sin\theta'.$$

Squaring, substituting in (1), and arranging, we have

$$\left\{\begin{aligned} &(a^2 \sin^2\theta - b^2 \cos^2\theta)\, x'^2 \\ &+ (a^2 \sin^2\theta' - b^2 \cos^2\theta')\, y'^2 \\ +\, 2\,&(a^2 \sin\theta \sin\theta' - b^2 \cos\theta \cos\theta')\, x'y' \end{aligned}\right\} = -a^2b^2. \tag{2}$$

From Art. 114, Cor. 2, we have

$$\tan^2\theta = \frac{b^2}{a^2} = \tan^2\theta';$$

from which we get

$$a^2\sin^2\theta - b^2\cos^2\theta = 0, \tag{3}$$

and

$$a^2\sin^2\theta' - b^2\cos^2\theta' = 0. \tag{4}$$

Also, from Art. 114, Cor. 2, we have

$$\sin\theta\sin\theta' = -\frac{b^2}{a^2+b^2},$$

and

$$\cos\theta\cos\theta' = \frac{a^2}{a^2+b^2};$$

therefore,

$$a^2\sin\theta\sin\theta' - b^2\cos\theta\cos\theta' = -\frac{2a^2b^2}{a^2+b^2}. \tag{5}$$

Substituting (3), (4), and (5) in (2), we get

$$-\frac{4a^2b^2}{a^2+b^2}x'y' = -a^2b^2;$$

or suppressing accents from the variables and reducing, we have

$$xy = \frac{a^2+b^2}{4}, \tag{6}$$

and putting m^2 for $\frac{a^2+b^2}{4}$, we have,

$$xy = m^2, \tag{7}$$

which is the equation required.

Cor.—The equation of the *conjugate* hyperbola, referred to the same axes, is (Art. 105)

$$xy = -m^2. \tag{8}$$

If we solve (7) for x, we get

$$x = \frac{m^2}{y},$$

which shows that as y increases, x diminishes, and when $y = \infty$, $x = 0$; that is, the curve approaches the axis of y, and finally touches it at an infinite distance from the centre.

Similarly, the curve approaches the axis of x, and finally touches it at an infinite distance from the centre.

Sch.—The second member of (7) is essentially positive, and of (4) essentially negative; hence, both x and y have the same sign in (7) and contrary signs in (8); therefore one branch of the given hyperbola lies wholly in the first angle and the other in the third; while one branch of the *conjugate* hyperbola lies wholly in the second and the other in the fourth angle. (See Fig. 96.)

In the case of *equilateral* hyperbolas (Art. 105, Sch. 2), the angle between the asymptotes, which (Art. 121, Sch.) is equal to $\sin^{-1}\frac{2ab}{a^2+b^2} = \sin^{-1} 1$, becomes a right angle; therefore, the *equilateral* hyperbola is also called the **Rectangular** hyperbola.

134. *To find the equation of the tangent at any point of an hyperbola referred to the asymptotes as axes.*

Let (x', y') and (x'', y'') be any two points, P and P′, on the curve. The equation of the secant through these points (Art. 26), is

$$y - y' = \frac{y' - y''}{x' - x''}(x - x'). \quad (1)$$

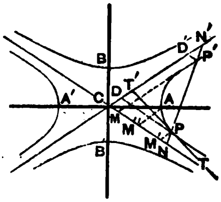

Fig. 97.

Since (x', y') and (x'', y'') are on the curve, we have (Art. 133),

$$x'y = m^2 = x''y'', \qquad \text{or} \qquad y'' = \frac{x'y'}{x''},$$

which in (1) gives $y - y' = -\frac{y'}{x''}(x - x')$, (2)

which is the equation of the *secant* to the hyperbola.

When the points become consecutive, we have $x'' = x'$; hence (2) becomes

$$y - y' = -\frac{y'}{x'}(x - x'). \tag{3}$$

Clearing (3) of fractions, transposing, and uniting, we have

$$x'y + y'x = 2x'y',$$

or

$$\frac{x}{x'} + \frac{y}{y'} = 2, \tag{4}$$

which is the equation of the tangent required.

COR. 1.—Making y and x successively $= 0$ in (4) we get

$$x = 2x' = \text{CT}, \quad \text{and} \quad y = 2y' = \text{CT}'.$$

Hence, P is the middle point of TT′; therefore, *the portion of the tangent included between the asymptotes is bisected at the point of contact.*

COR. 2.—From Cor. 1, we have,

$$\text{CT} \times \text{CT}' = 4x'y' = a^2 + b^2 \text{ (Art. 133)}.$$

That is, *the rectangle of the intercepts cut off upon the asymptotes by any tangent is constant, and equal to the sum of the squares on the semi-axes.*

COR. 3.—The area of the triangle TCT′, Fig. 97, is

$$= \tfrac{1}{2}\text{CT} \times \text{CT}' \sin \text{TCT}'$$

$$= 2x'y' \times \frac{2ab}{a^2 + b^2} \text{ (Cor. 1, and Art. 121, Sch.)},$$

$$= \frac{a^2 + b^2}{2} \times \frac{2ab}{a^2 + b^2} \text{ (Art. 133)}$$

$$= ab = \textit{constant}.$$

Therefore, *the triangle included between any tangent and the asymptotes is constant, and equal to the rectangle of the semi-axes.*

135. *To prove that the intercepts of a secant between the hyperbola and its asymptotes are equal.*

In equation (2) of Art. 134, make $y = 0$, and get

$$x = x'' + x'$$
$$= \text{CN (Fig. 97)}.$$

Hence, $\text{CN} - x' = x''$,

or $\text{M'N} = \text{D'P'}$;

therefore, $\text{NP} = \text{N'P'}$;

that is, *the intercepts of the secant are equal.*

Sch.—This proposition affords a convenient method of constructing the curve. If the axes are given, construct the rectangle on them, the diagonals of which are the asymptotes. Then through the extremity of the transverse axis, draw a right line intercepted by the asymptotes; lay off on this line from one asymptote a distance equal to the extremity of the axis from the other asymptote; the point thus found will be a point of the curve. In this manner, find any number of points, and draw a line through them; *this will be the required curve.*

136. *To prove that the parallelogram formed by drawing lines from any point of an hyperbola parallel to and terminating in the asymptotes, is equal to one-eighth the rectangle on the axes.*

Call ϕ the angle TCT′ (Fig. 97); the area of CM′PD

$$= x'y' \sin \phi$$
$$= \frac{a^2 + b^2}{4} \times \frac{2ab}{a^2 + b^2} \text{ (Art. 133, and Art. 121, Sch.)}$$
$$= \tfrac{1}{2}ab = \tfrac{1}{8}(2a \cdot 2b),$$

which proves the proposition.

137. *To find the equations of two conjugate diameters of an hyperbola referred to its asymptotes.*

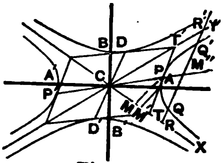

Fig. 98.

The diameter which passes through the origin and the point P (x', y') is represented (see Art. 26, Cor. 4) by

$$y = \frac{y'}{x'}x,$$

or

$$\frac{x}{x'} - \frac{y}{y'} = 0. \qquad (1)$$

The diameter conjugate to this one, CD, is parallel to the tangent at (x', y'), and therefore (Art. 134, Eq. 4) its equation is

$$y = -\frac{y'}{x'}x,$$

or

$$\frac{x}{x'} + \frac{y}{y'} = 0. \qquad (2)$$

Cor.—When the diameters PP′ and DD′ become the axes, AA′ and BB′, we have, since the axes bisect the angle between the asymptotes,

$$\text{CM}' = \text{M}'\text{A}, \qquad \text{or} \qquad x' = y';$$

therefore (1) and (2) become

$$x - y = 0, \qquad \text{and} \qquad x + y = 0,$$

which are the equations of the axes referred to the asymptotes.

138. *Given the co-ordinates of the extremity of a diameter, to find those of the extremity of its conjugate.*

Let (x', y') be the point P (Fig. 98), and (x'', y'') the point D.

The equation of DD′ (Art. 137) is

$$\frac{x}{x'} + \frac{y}{y'} = 0. \qquad (1)$$

The equation of the conjugate hyperbola (Art. 105) is

$$a^2y^2 - b^2x^2 = a^2b^2. \tag{2}$$

Eliminating between (1) and (2), we get

$$x'' = \mp x', \qquad y'' = \pm y'. \tag{3}$$

Cor. 1.—The equation of the tangent at P (x', y') (Art. 134) is

$$\frac{x}{x'} + \frac{y}{y'} = 2. \tag{4}$$

The equation of the tangent at D (x'', y''), the extremity of the conjugate diameter (Art. 134) is

$$\frac{x}{x''} + \frac{y}{y''} = 2,$$

or from (3),

$$\frac{x}{x'} - \frac{y}{y'} = -2. \tag{5}$$

Adding (4) and (5), we get $x = 0$ as the locus of the intersection of the tangents (4) and (5), which is the equation of the axis of y, or the asymptote CR′. Therefore, *tangents at the extremities of conjugate diameters meet on the asymptotes.*

Cor. 2.—Since T′ is a vertex of the parallelogram formed on the conjugate diameters PP′ and DD′, we have

$$\text{PT}' = \text{CD};$$

therefore,

$$\text{TT}' = 2\text{PT}' = \text{DD}';$$

that is, *the portion of the tangent at any point of an hyperbola, included between the asymptotes, is equal to the diameter conjugate to that which passes through the point of contact.*

139. *If a chord be drawn parallel to any diameter, it will be bisected by the conjugate diameter produced.*

Let QQ′ be drawn parallel to DD′ (Fig. 98); then will it be bisected at M″ by CP produced.

Since QQ′ is parallel to DD′, its equation will differ from that of DD′ only by a constant term; therefore [Art. 138, (1)]

$$\frac{x}{x'} + \frac{y}{y'} = c \qquad (1)$$

is the equation of QQ′.

Combine (1) with the equation of PP′ (Art. 137), which is

$$\frac{x}{x'} - \frac{y}{y'} = 0, \qquad (2)$$

and we get $x = \frac{1}{2}cx'$, $y = \frac{1}{2}cy'$,

as the co-ordinates of M″. But from (1) we have

$$\text{CR} = cx', \quad \text{and} \quad \text{CR}' = cy';$$

therefore M″ is the middle point of RR′. But (Art. 135),

$$\text{RQ} = \text{R}'\text{Q}';$$

therefore, QM″ = M″Q′, *which proves the proposition.*

EXAMPLES.

1. Find the axes of the hyperbola whose equation is $3y^2 - 2x^2 + 12 = 0$; also the eccentricity of the given and the conjugate hyperbola, and the parameter.

Ans. $a = \sqrt{6}$, $b = 2$; $e = \sqrt{\frac{5}{3}}$; $e' = \sqrt{\frac{5}{2}}$; $2p = \frac{8}{\sqrt{6}}$.

2. Find the intersection of the hyperbola $3y^2 - 2x^2 + 12 = 0$ and the circle $x^2 + y^2 = 16$. *Ans.* $(\pm 2\sqrt{3}, \pm 2)$.

3. Find whether the line $y = \frac{3}{4}x$ cuts the hyperbola $5y^2 - 2x^2 = -15$, or its conjugate.

Ans. It cuts the conjugate.

4. Find the equation of an hyperbola of given transverse axis, whose vertex bisects the distance between the centre and the focus. *Ans.* $y^2 - 3x^2 = -3a^2$.

5. If the ordinate MP (Fig. 95) of an hyperbola be produced to Q, so that MQ = F′P, find the locus of Q.

Ans. A right line.

6. If an ellipse and an hyperbola have the same foci, prove that their tangents at the points of intersection are at right angles. (See Art. 75, Cor. 2, and Art. 107, Cor.)

7. Find the condition that the line $\left(\frac{x}{m} + \frac{y}{n} = 1\right)$ shall touch the hyperbola $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\right)$. *Ans.* $\frac{a^2}{m^2} - \frac{b^2}{n^2} = 1$.

[To obtain this, compare $\frac{x}{m} + \frac{y}{n} = 1$ with equation of tangent (Art. 107), which is

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1 \text{ (Art. 105)},$$

and we have $\quad \frac{x'}{a} = \frac{a}{m} \quad$ and $\quad \frac{y'}{b} = -\frac{b}{n},$

which in equation of curve gives the answer.]

8. Find where the tangents from the foot of the directrix will meet the hyperbola, and what angle they will make with the transverse axis.

Ans. The extremity of the *latus rectum ;* $\tan^{-1} \pm e$.

9. Find the angle included between the asymptotes of the hyperbola $16y^2 - 9x^2 = -25$. *Ans.* 73° 44′.

10. Find the perpendicular from the focus of any hyperbola to its asymptotes. *Ans.* The semi-conjugate axis.

11. If 3AC = 2CF′ (Fig. 95), find the inclination of the asymptotes to the transverse axis. *Ans.* $\tan^{-1} \frac{\sqrt{5}}{2}$.

12. If the asymptotes of the hyperbola are axes, show that the equation of one directrix is $x + y - a = 0$.

[See Art. 137, Cor.]

13. Prove that if a circle be described with the focus of an hyperbola for its centre and with a radius equal to the semi-conjugate axis, it will touch the asymptotes in the points where they are cut by the directrix.

14. Prove that the radius of a circle which touches an hyperbola and its asymptotes is equal to that part of the latus rectum produced which is intercepted between the curve and the asymptote.

15. Find the length of the normal NP and of RP (Fig. 88). [See Art. 108, Cor. 1.] *Ans.* $\text{NP} = \frac{bb'}{a}$, $\text{RP} = \frac{ab'}{b}$.

16. Prove that the product of the two perpendiculars let fall from any point of an hyperbola upon the asymptotes is constant and equal to $\frac{a^2b^2}{a^2+b^2}$.

17. Tangents to an hyperbola are drawn from any point on either branch of the conjugate curve; prove that their chord of contact touches the opposite branch of the conjugate curve.

[Take the diameter passing through the *point* for axis of y, and the conjugate diameter for axis of x; equation of chord of contact is

$$\frac{xx'}{a'^2} - \frac{yy'}{b'^2} = 1,$$

which soon reduces to $y = b'$; $\therefore$ etc.]

18. In any equilateral hyperbola, let $\phi =$ the inclination of a diameter, passing through any point P, and $\phi' =$ that of the polar of P, the transverse axis being the axis of x; prove that $\tan\phi \tan\varphi' = 1$.

[Equation of diameter is $y = \frac{y'}{x'}x$; $\therefore$ $\frac{y'}{x'} = \tan\phi$; polar of P is $xx' - yy' = a^2$; $\therefore$ $\frac{x'}{y'} = \tan\phi'$; $\therefore$ etc.]

19. Prove that the middle points of a series of parallels intercepted between an hyperbola and its conjugate, lie on the curve whose equation is

$$4\left(\frac{x^2}{a'^2} - \frac{y^2}{b'^2}\right) = \frac{b'^2}{y^2}.$$

[Take for axis of y the diameter parallel to the lines, and for axis of x the conjugate diameter.]

20. Between the sides of a given angle ϕ, a right line moves so as to inclose a triangle of constant area $= k^2$; prove that the locus of the centre of gravity of the triangle is the hyperbola whose equation is $9xy \sin \phi = 2k^2$.

[Take the sides of the angle for the axes.]

21. A tangent at the extremity of the latus rectum of an hyperbola meets any ordinate MP produced in R; prove that FP = MR, where F is the focus through which the latus rectum passes.

22. If from a point P in an hyperbola PK be drawn parallel to the transverse axis, cutting the asymptotes in I and K, prove that PK × PI $= a^2$; or, if parallel to the conjugate, PK × PI $= b^2$.

[Combine equation of line through P (x', y') with equations of asymptotes, etc.]

CHAPTER VIII.

GENERAL EQUATION OF THE SECOND DEGREE.

140. It has been shown (Art. 23) that every equation of the first degree between two variables is the equation of a right line. We have seen that the equations of the circle, parabola, ellipse, and hyperbola are all of the second degree. We shall now show that *every* equation of the second degree between two variables is the equation of a circle, a parabola, an ellipse, an hyperbola, or a right line, two right lines, or a point.

141. The most general form of the equation of the second degree is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \qquad (1)$$

where a, b, c, d, e, f are all constants.

Five relations between the coefficients are sufficient to determine a locus of the second degree, although (1) contains *six* constants. The nature of the locus depends, not on the *absolute magnitude* of the coefficients, but on their *mutual ratios*, for if we multiply or divide (1) by any constant, it will still clearly represent the same locus. We may, therefore, divide (1) by f, so as to make the absolute term $= 1$, and there will then remain but five constants to be determined.

If the locus passes through the origin, $f = 0$ (see Art. 41, Cor. 2), and (1) becomes

$$ax^2 + bxy + cy^2 + dx + ey = 0, \qquad (2)$$

which is the equation of the locus when it passes through the origin.

If the origin of co-ordinates be taken at the *centre* of the locus (Art. 71, Sch.), for every point (x', y') whose co-ordinates satisfy the equation, there will be a corresponding point $(-x', -y')$ whose co-ordinates also will satisfy the equation; hence, *when the centre is the origin, the equation will not be altered by writing* $-x$, $-y$ *for* x, y; therefore, the terms of the first degree must vanish from it. In order, then, to find the *centre* of the locus, we must transfer the origin to a point (x', y'), and then see what values of x', y' will make the new coefficients of x and y vanish. These values of x' and y' will be the co-ordinates of the *centre* of the locus referred to the original axes. In the following transformations, we shall suppose the co-ordinate axes rectangular; for if they were oblique, we might transform the equation to one in which the axes were rectangular, without affecting the *degree* or *form* of the equation.

FIRST TRANSFORMATION.

142. The object of this transformation is to remove from

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \qquad (1)$$

the terms involving the *first* power of x and y. To do this we transform (1) to parallel axes passing through a new origin (x', y').

The formulæ for transformation to parallel axes through (x', y') are (Art. 33), $x = x' + x$, $y = y' + y$, where x' and y' are put for m and n. Substituting these values for x and y in (1), and arranging the terms of the resulting equation, we have

$$ax^2 + bxy + cy^2 + \left.\begin{array}{r}2ax'\\ +\ by'\\ +\ d\end{array}\right| x + \left.\begin{array}{r}2cy'\\ +\ bx'\\ +\ e\end{array}\right| y + \left.\begin{array}{r}ax'^2\\ +\ bx'y'\\ +\ cy'^2\\ +\ dx'\\ +\ ey'\\ +\ f\end{array}\right| = 0, \qquad (2)$$

or
$$ax^2 + bxy + cy^2 + d'x + e'y + f' = 0, \tag{3}$$

from which we see that the coefficients of x^2, xy, and y^2 are, as before, a, b, c; that

the new d is $d' = 2ax' + by' + d$;
the new e is $e' = 2cy' + bx' + e$;
the new f is $f' = ax'^2 + bx'y' + cy'^2 + dx' + ey' + f$.

Hence, *if the equation of a locus of the second degree be transformed to parallel axes through a new origin, the coefficients of the highest powers of the variables will remain unchanged, while the new absolute term will be the result of substituting in the original equation the co-ordinates of the new origin.*

Putting the coefficients of x and y in (2) equal to 0, we have

$$2ax' + by' + d = 0, \tag{4}$$

$$2cy' + bx' + e = 0, \tag{5}$$

which are the equations for the centre of (1).

Equations (4) and (5) may thus be obtained: *For* (4) *take only those terms of* (1) *which involve x; multiply each term by the exponent of x in it, and diminish that exponent by unity.* Equation (5) may be obtained similarly by substituting y for x in the above rule. Thus, the equations for the centre of the locus represented by

$$4x^2 + 3xy + 2y^2 - 14y + 17 = 0$$

are $8x + 3y = 0$ and $3x + 4y - 14 = 0$.

Sch.—Solving (4) and (5) for x' and y', we find them to be

$$x' = \frac{2cd - be}{b^2 - 4ac}, \tag{6}$$

and
$$y' = \frac{2ae - bd}{b^2 - 4ac}, \tag{7}$$

which are the co-ordinates of the *centre* with reference to the old axes.

It is plain that these values of x' and y' will always be finite, except when $b^2-4ac=0$, in which case they will be infinite. Hence, loci of the second degree may be divided into two classes: I, those which *have a centre;* II, those which in general *have not a centre,* or rather, whose centre is infinitely distant. The first are often called **Central Curves**, while the second are called **Non-central Curves.** We shall first consider the case of central loci.

Substituting (4) and (5) in (2), and representing the absolute term by f', for shortness, we have

$$ax^2+bxy+cy^2+f'=0. \tag{8}$$

We see that if (8) is satisfied by any values, x' and y' for x and y, it is also satisfied by the values $-x'$ and $-y'$. Hence, the origin of co-ordinates in (8) is *the centre* of the locus which (1) or (8) represents.

SECOND TRANSFORMATION.

143. The object of this transformation is to remove from

$$ax^2+bxy+cy^2+f'=0 \tag{1}$$

the term involving xy, and leave (1) in the form

$$a'x^2+c'y^2+f'=0,$$

where if any value be given to one of the variables, the other will have two equal values, with contrary signs.

To effect this transformation, we revolve the axes of co-ordinates through the angle θ till they coincide with the axes of the locus. The formulæ for this transformation (Art. 35, Cor. 3), are

$$x=x'\cos\theta-y'\sin\theta,$$
$$y=x'\sin\theta+y'\cos\theta.$$

Substituting these values for x and y in (1), and arranging the terms, we have

$$\begin{array}{r|lr|lr|l}
a\cos^2\theta & x'^2 & -2a\sin\theta\cos\theta & x'y' & +a\sin^2\theta & y'^2+f'=0. \\
+b\sin\theta\cos\theta & & +b\cos^2\theta & & -b\sin\theta\cos\theta & \\
+c\sin^2\theta & & -b\sin^2\theta & & +c\cos^2\theta & \qquad (2) \\
 & & +2c\sin\theta\cos\theta & & &
\end{array}$$

If we equate the new coefficient of $x'y'$ to 0, we obtain

$$2(c-a)\sin\theta\cos\theta + b(\cos^2\theta - \sin^2\theta) = 0,$$

or
$$(c-a)\sin 2\theta + b\cos 2\theta = 0;$$

therefore,
$$\tan 2\theta = \frac{b}{a-c}, \qquad (3)$$

from which we may determine the angle θ through which the co-ordinate axes must be turned to remove the term containing xy.

As the tangent of an angle may have any value, positive or negative, from 0 to ∞, it follows that (3) will always give real values for 2θ; that is, there are two real lines at right angles with each other to which when the locus is referred, the term involving xy vanishes.

Substituting (3) in (2), we have, for the required transformation of (1),

$$\left\{\begin{array}{r} (a\cos^2\theta + b\sin\theta\cos\theta + c\sin^2\theta)\,x'^2 \\ +\ (a\sin^2\theta - b\sin\theta\cos\theta + c\cos^2\theta)\,y'^2 \\ +f' \end{array}\right\} = 0, \qquad (4)$$

or, omitting the accents from the variables, and writing a' and c' for the coefficients of x^2 and y^2, we have

$$a'x^2 + c'y^2 + f' = 0, \qquad (5)$$

which is the equation of the locus referred to its centre and axes.

To find the values of a' and c' in (5), we have

$$\begin{aligned} a' &= a\cos^2\theta + b\sin\theta\cos\theta + c\sin^2\theta \text{ [from (4)]} \\ &= \tfrac{1}{2}[a\cos^2\theta + a(1-\sin^2\theta) + c\sin^2\theta + c(1-\cos^2\theta) \\ &\qquad + 2b\sin\theta\cos\theta] \\ &= \tfrac{1}{2}[a + c + (a-c)\cos 2\theta + b\sin 2\theta]. \qquad (4)' \end{aligned}$$

Similarly, $c' = \frac{1}{2}[a+c-(a-c)\cos 2\theta - b\sin 2\theta]$. (5)′

From Trigonometry we have

$$\cos 2\theta = \frac{1}{\sqrt{1+\tan^2 2\theta}} = \frac{a-c}{\sqrt{b^2+(a-c)^2}}$$

[from (3)]; also

$$\sin 2\theta = \sqrt{1-\cos^2 2\theta} = \frac{b}{\sqrt{b^2+(a-c)^2}}.$$

Substituting these values of $\cos 2\theta$ and $\sin 2\theta$ in (4)′ and (5)′, we get

$$a' = \tfrac{1}{2}\left[a+c+\frac{(a-c)^2+b^2}{\sqrt{b^2+(a-c)^2}}\right]$$

$$= \tfrac{1}{2}\left[a+c+\sqrt{b^2+(a-c)^2}\right] \qquad (6)$$

also,
$$c' = \tfrac{1}{2}\left[a+c-\frac{(a-c)^2+b^2}{\sqrt{b^2+(a-c)^2}}\right]$$

$$= \tfrac{1}{2}\left[a+c-\sqrt{b^2+(a-c)^2}\right]. \qquad (7)$$

Hence, we see that the general equation of the second degree given in (1) of Art. 141 can always be transformed to the form given in (5), *provided* that it is not subject to the condition $b^2-4ac=0$ (Art. 142, Sch.).

COR. 1.—Multiplying (6) and (7) together, we have

$$a'c' = \tfrac{1}{4}[(a+c)^2-b^2-(a-c)^2] = \tfrac{1}{4}(4ac-b^2).$$

Hence, if a' and c' have *like* signs, $4ac-b^2$ will be *positive*, or b^2-4ac will be *negative;* but if a' and c' have *unlike* signs, b^2-4ac will be *positive.*

COR. 2.—When $b^2-4ac<0$, a' and c' have the same sign (Cor. 1); if f' have a *contrary* sign from a' and c', (5) becomes

$$\frac{a'}{f'}x^2+\frac{c'}{f'}y^2=1, \qquad (8)$$

which is the equation of an ellipse [Art. 71, (3)] whose axes are $\sqrt{\frac{f'}{a'}}$ and $\sqrt{\frac{f'}{c'}}$.

If $c' = a'$, (8) becomes

$$x^2 + y^2 = \frac{f'}{a'}, \tag{9}$$

which is the equation of a circle whose radius is $\sqrt{\frac{f'}{a'}}$.

If $f' = 0$, (8) becomes

$$a'x^2 + c'y^2 = 0,$$

which is the equation of the two imaginary right lines

$$x\sqrt{a'} + y\sqrt{-c'} = 0, \quad \text{and} \quad x\sqrt{a'} - y\sqrt{-c'} = 0,$$

which meet in the real point $x = 0$, $y = 0$; or it is the equation of the origin, or the ellipse diminished indefinitely.

If f' have the *same* sign as a' and c', (5) becomes

$$\frac{a'}{f'}x^2 + \frac{c'}{f'}y^2 = -1,$$

which cannot be satisfied by any real value of x and y; therefore the locus is *imaginary*.

Hence, *if $b^2 - 4ac < 0$, the general equation of the second degree between two variables represents an ellipse, a circle, a point, or an imaginary locus.*

COR. 3.—When $b^2 - 4ac > 0$, a' and c' have unlike signs (Cor. 1). Suppose c' and f' to be positive, and a' to be negative; (5) becomes

$$\frac{a'}{f'}x^2 - \frac{c'}{f'}y^2 = 1, \tag{10}$$

which is the equation of an hyperbola [Art. 102, (3)] whose axes are $\sqrt{\frac{f'}{a'}}$ and $\sqrt{\frac{f'}{c'}}$.

If a' and f' are positive, and c' negative, (5) becomes

$$\frac{a'}{f'}x^2 - \frac{c'}{f'}y^2 = -1, \qquad (11)$$

which is the equation of an hyperbola [Art. 105, (1)] conjugate to (10).

If $f' = 0$, (10) or (11) becomes

$$a'x^2 - c'y^2 = 0,$$

which is the equation of the two lines

$$y = \pm x\sqrt{\frac{a'}{c'}}$$

intersecting at the origin.

If $c' = a'$, (10) and (11) become

$$x^2 - y^2 = \frac{f'}{a'}, \quad \text{and} \quad x^2 - y^2 = -\frac{f'}{a'},$$

which are equilateral hyperbolas [Art. 105, (2) and (3)].

Hence, *if $b^2 - 4ac > 0$, the general equation of the second degree between two variables represents an hyperbola or its conjugate, an equilateral hyperbola, or two right lines intersecting each other.*

144. We have shown (Art. 142) that the coefficients of the first three terms of the general equation of the second degree between two variables are not altered by a transfer of the origin; we shall now show that when the axes are turned through an angle θ, and the new coefficients of the first three terms are denoted by a', b', c', we have the relations $a' + c' = a + c$ and $b'^2 - 4a'c' = b^2 - 4ac$.

From (2) of Art. 143, we have

$$\begin{aligned} a' &= a\cos^2\theta + b\sin\theta\cos\theta + c\sin^2\theta \\ &= \tfrac{1}{2}[a+c+(a-c)\cos 2\theta + b\sin 2\theta] \text{ [from (4)']}. \qquad (1) \\ b' &= 2(c-a)\sin\theta\cos\theta + b(\cos^2\theta - \sin^2\theta) \\ &= (c-a)\sin 2\theta + b\cos 2\theta. \qquad (2) \end{aligned}$$

$$c' = a\sin^2\theta - b\sin\theta\cos\theta + c\cos^2\theta$$
$$= \tfrac{1}{2}\{a+c-[(a-c)\cos 2\theta + b\sin 2\theta]\} \text{ [from (5)']}. \quad (3)$$

Adding (1) and (3), we get

$$a' + c' = a + c. \quad (4)$$

Also, from (1), (2), and (3), we have

$$b'^2 - 4a'c' = \left\{ \begin{matrix} [(c-a)\sin 2\theta + b\cos 2\theta]^2 \\ -\{(a+c)^2 - [(a-c)\cos 2\theta + b\sin 2\theta]^2\} \end{matrix} \right\}$$

$$= \left\{ \begin{matrix} (a-c)^2(\sin^2 2\theta + \cos^2 2\theta) \\ + b^2(\cos^2 2\theta + \sin^2 2\theta) - (a+c)^2 \end{matrix} \right\}$$

$$= (a-c)^2 + b^2 - (a+c)^2;$$

that is,
$$b'^2 - 4a'c' = b^2 - 4ac. \quad (5)$$

Thus, the expression $b^2 - 4ac$ has the same value whether it be formed from the coefficients of the *general equation* of the second degree, as given in (1) of Art. 141, or after one or both transformations have been made, as in (8) of Art. 142, or (5) of Art. 143.

145. *To sum up briefly:*

1st. *In order to reduce the equation of a central locus to parallel axes through its centre, we have the following directions:*

1. The coefficients of the first three terms remain unaltered (Art. 142).

2. The co-ordinates of the centre of the locus are given by (6) and (7) of Art. 142, Sch.

3. The absolute term is replaced by a new one, which is the result of substituting in the original equation the co-ordinates of the centre (Art. 142).

The equation is now reduced to the form

$$ax^2 + bxy + cy^2 + f' = 0 \text{ [Art. 142, (8)]}, \quad (1)$$

where the origin is at the centre of the locus.

2d. *To reduce* (1) *to the form* $a'x^2 + c'y^2 + f' = 0$ *by turning the axes through the angle* $\theta = \frac{1}{2}\tan^{-1}\frac{b}{a-c}$. (Art. 143.)

4. The coefficients a' and c' are given in (6) and (7) of Art. 143.

5. The absolute term, f', remains unaltered [Art. 143, (2)].

The equation is now reduced to the form

$$a'x^2 + c'y^2 + f' = 0 \text{ [Art. 143, (5)].} \qquad (2)$$

146. We shall now consider the case in which

$$b^2 - 4ac = 0.$$

We saw (Art. 142, Sch.) that in this case the centre was infinitely distant, or, in other words, that there was *no centre*. We cannot, therefore, remove the terms dx and ey from the general equation by changing the origin to the centre, as we did in Art. 142; but we can remove the term xy from the equation by turning the axes through the angle θ, as we did in Art. 143, where θ is obtained from (3) of Art. 143.

Substituting $x'\cos\theta - y'\sin\theta$ for x, and $x'\sin\theta + y'\cos\theta$ for y in (1) of Art. 142, and arranging as in (2) of Art. 143, we have

$$\left(\begin{array}{r} (a\cos^2\theta + b\sin\theta\cos\theta + c\sin^2\theta)\,x'^2 \\ (-2a\sin\theta\cos\theta + b\cos^2\theta - b\sin^2\theta + 2c\sin\theta\cos\theta)\,x'y' \\ (+\,a\sin^2\theta - b\sin\theta\cos\theta + c\cos^2\theta)\,y'^2 \\ (+\,d\cos\theta + e\sin\theta)\,x' \\ (+\,e\cos\theta - d\sin\theta)\,y' \\ +f \end{array}\right) = 0. \qquad (1)$$

Now, for $\tan 2\theta = \frac{b}{a-c}$, the term containing $x'y'$ in (1) vanishes, by Art. 143, (2); and if we denote the coefficients of x'^2, y'^2, x', y', by a', c', d', e', (1) becomes

$$a'x'^2 + c'y'^2 + d'x' + e'y' + f = 0, \qquad (2)$$

where a' and c' have the values given in Art. 143, (6) and (7), and d' and e' have the values given in (1); that is,

$$d' = d\cos\theta + e\sin\theta, \qquad e' = e\cos\theta - d\sin\theta. \qquad (2)'$$

Now (Art. 143, Cor. 1), $a'c' = \frac{1}{4}(4ac - b^2)$, which, by the present hypothesis, is equal to 0; therefore, $a'c' = 0$, and hence either a' or c' must equal 0. We shall suppose that $a' = 0$, which reduces (2), by omitting accents, to

$$c'y^2 + d'x + e'y + f = 0. \qquad (3)$$

Remark.—If we were to suppose $c' = 0$, instead of $a' = 0$, the equation would represent the same form of locus that (3) represents, except that the locus would be situated with respect to the axes of x and y just as that of (3) is situated with respect to the axes of y and x respectively.

Now transform the origin to a point (x', y'), by putting $x' + x$ for x, and $y' + y$ for y in (3), and it becomes,

$$c'y^2 + d'x + (2c'y' + e')\,y + (c'y'^2 + d'x' + e'y' + f) = 0. \qquad (4)$$

Equating the coefficient of y and the absolute term $= 0$, in (4), we have

$$2c'y' + e' = 0, \quad \text{or} \quad y' = -\frac{e'}{2c'}; \qquad (5)$$

$$c'y'^2 + d'x' + e'y' + f = 0, \quad \text{or} \quad x' = \frac{e'^2 - 4c'f}{4c'd'}; \qquad (6)$$

and (4) becomes

$$c'y^2 + d'x = 0, \quad \text{or} \quad y^2 = -\frac{d'}{c'}x, \qquad (7)$$

which is the equation of a parabola [Art. 53, (2)], in which the axis of x is the *axis of the curve*, and the origin of coordinates is at *the vertex*. If d' and c' have the same sign, the curve is to the *left* of the origin; and if d' and c' have unlike signs, the curve is to the *right* of the origin (Art. 53, Cor. 2).

Cor.—If $d' = 0$ in (3), the equation becomes

$$c'y^2 + e'y + f = 0, \tag{8}$$

or
$$y = \frac{-e' \pm \sqrt{e'^2 - 4c'f}}{2c'}, \tag{9}$$

which represents (Art. 23) two right lines parallel to the new axis of x, which are *real and different, real and coincident,* or *imaginary,* according as

$$e'^2 - 4c'f >, \quad =, \quad < 0.$$

Hence, when $b^2 - 4ac = 0$, *the general equation of the second degree between two variables represents a parabola, two parallel right lines, two coincident right lines, or two imaginary parallel right lines.*

Sch.—The results of the foregoing articles, as determining the species of the locus may be summed up as follows:

The general equation of the second degree between two variables always represents a parabola, an ellipse, an hyperbola, or some one of their limiting cases.

$b^2 - 4ac = 0$ represents the parabola.
$b^2 - 4ac < 0$ " " ellipse.
$b^2 - 4ac > 0$ " " hyperbola.

EXAMPLES.

1. Determine the species and situation of the locus whose equation is

$$5x^2 + 2xy + 5y^2 - 12x - 12y = 0, \tag{1}$$

and transform the equation (Arts. 142, 143) to its axes, and illustrate each transformation by a figure.

Since the absolute term is wanting, the locus passes through the origin (Art. 141).

Here $b = 2$, $a = 5$, $c = 5$; hence, $b^2 - 4ac = -96$, < 0. Therefore the locus is an ellipse.

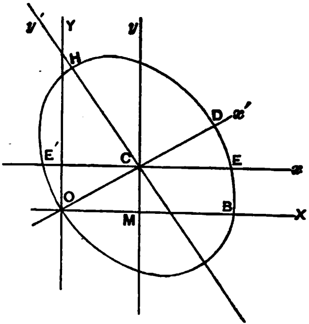

Fig. 99.

1st. By Art. 145, (2), the co-ordinates of the centre are $x = y = 1$. That is, the centre of the ellipse is at the point (1, 1).

If then we transform the origin from O to C, the centre, so that OM = CM = 1, and the new axes of co-ordinates, Cx and Cy, are parallel to the old ones, OX and OY, the transformed equation is found to be

$$5x^2 + 2xy + 5y^2 - 12 = 0. \tag{2}$$

2d. By Art. 145, (4), we have

$$a' = \tfrac{1}{2}\left(5 + 5 + \sqrt{4 + 0}\right) \text{ [Art. 143, (6)]} = 6;$$

$$c' = \tfrac{1}{2}\left(5 + 5 - \sqrt{4 + 0}\right) = 4.$$

$$\tan 2\theta = \frac{b}{a - c} \text{ [Art. 143, (3)]} = \frac{2}{0} = \infty;$$

hence, $\theta = 45°$; that is, the new axis of x is inclined to the original axis of x at an angle of 45°.

Hence, if the axes Cx and Cy are turned through 45°, (2) becomes

$$6x^2 + 4y^2 - 12 = 0,$$

or

$$\tfrac{1}{2}x^2 + \tfrac{1}{3}y^2 = 1, \tag{3}$$

which is the equation of an ellipse referred to its axes, the axis of x coinciding with the minor axis, and the axis of y with the major axis, the semi-axes being $\sqrt{3}$ and $\sqrt{2}$; therefore the major axis of the ellipse is inclined to the original axis of x at an angle of 135°.

To construct the figure, let OX and OY be the original axes; locate the new origin C at (1, 1), and draw the second set of axes, Cx and Cy, parallel to the old; then, as $\theta = 45°$, draw Cx', making with Cx an angle of 45°, and Cy' perpendicular to Cx'; lay off $CD = \sqrt{2}$ and $CH = \sqrt{3}$, as these are the semi-axes. The rest of the construction is as in Art. 72, giving us Fig. 99.

To find where the locus cuts the original axis of x, make $y = 0$ in (1), and get, after dividing by 5,

$$x^2 - \tfrac{12}{5}x = 0,$$

from which we have $x = 0$ and $x = 2\tfrac{2}{5}$ as the points O and B.

To find where the locus cuts the second axis of x, make $y = 0$ in (2) and get $x^2 = \tfrac{12}{5}$, from which we have

$$x = \pm\sqrt{\tfrac{12}{5}},$$

as the points E and E'.

To find where it cuts the new axis of x, make $y = 0$ in (3), and get $x^2 = 2$, or $x = \pm\sqrt{2}$, as the points D and O.

2. Find the species and situation of the locus

$$2xy - x + 1 = 0, \qquad (1)$$

and transform and construct as in Ex. 1.

Here $b = 2$, $a = 0$, $c = 0$; $\therefore\ b^2 - 4ac = 4, > 0$; hence the locus is an hyperbola.

1st. By Art. 145, (2), the co-ordinates of the centre are $x = 0$, $y = \tfrac{1}{2}$.

Now transform to parallel axes through the centre C $(0, \tfrac{1}{2})$, and (1) becomes

$$2xy + 1 = 0. \qquad (2)$$

2d. By Art. 145, (4), we have

$$a' = \tfrac{1}{2}\left(0 + \sqrt{4}\right) = 1;$$

$$c' = -1 \quad [\text{Art. 143, (6) and (7)}];$$

$$\tan 2\theta = \frac{2}{0} = \infty; \quad \therefore\ \theta = 45^\circ;$$

that is, the new axis of x (Cx') is inclined to the old axis of x at an angle of 45°.

Therefore, turning the axes Cx and Cy through 45°, (2) becomes

$$x^2 - y^2 + 1 = 0, \tag{3}$$

which is the equation of an equilateral hyperbola referred to its axes. The form of (3) shows that the axis of x coincides with the conjugate axis, and the axis of y with the transverse axis (Art. 105, Sch. 2); therefore it is the conjugate or y hyperbola (Art. 105), the semi-axes being 1; and hence the transverse axis of the hyperbola is inclined to the original axis of x at an angle of 135°.

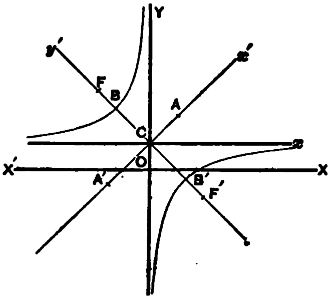

Fig. 100.

To construct the figure, let OX and OY be the old axes; locate the new origin C at $(0, \frac{1}{2})$, and draw the second set of axes Cx and Cy parallel to the first; then, as $\theta = 45°$, draw Cx' making with Cx an angle of 45°, and Cy' perpendicular to Cx'; lay off CA and $\text{CB} = 1$, as these are the semi-axes. The rest of the construction is like that in Art. 103, giving us Fig. 100.

To find where the locus cuts the original axis of x, make $y = 0$ in (1), and get $x = 1$.

The form of (2) shows it to be the equation of the conjugate hyperbola referred to its asymptotes (Art. 133), for (8) of Art. 133 is

$$xy = -\frac{a^2 + b^2}{4},$$

which in the present example becomes

$$xy = -\frac{1^2 + 1^2}{4} = -\frac{1}{2}.$$

3. Find the species and situation of the locus

$$x^2 - 2xy + y^2 - 8x + 16 = 0, \qquad (1)$$

and transform and construct.

$$b^2 - 4ac = 0;$$

therefore the locus is a parabola.

The transformation is effected by Art. 146.

$$\tan 2\theta = \frac{b}{a-c}$$

$$= -\infty;$$

$$\therefore \quad \theta = 45°.$$

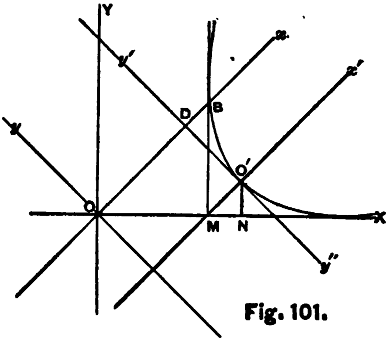

Fig. 101.

Hence the new axis of x (Ox) is inclined to the old at an angle of 45°.

$$c' = \tfrac{1}{2}(1 + 1 + \sqrt{4}) = 2 \text{ [Art. 143, (7)]};$$

[here we take the *minus* value of $\sqrt{b^2 + (a-c)^2}$ because we squared -2 to get b^2].

$$d' = \tfrac{1}{2}(-8\sqrt{2} + 0) = -4\sqrt{2} \text{ [Art. 146, (2)']}.$$

$$e' = \tfrac{1}{2}\sqrt{2}\,(8) \qquad = \quad 4\sqrt{2} \text{ [Art. 146, (2)']}.$$

Therefore, turning the axes OX and OY through 45°, (1) becomes

$$2y^2 - 4\sqrt{2}x + 4\sqrt{2}y + 16 = 0 \text{ [Art. 146, (3)]},$$

or $$y^2 - 2\sqrt{2}x + 2\sqrt{2}y + \ 8 = 0. \qquad (2)$$

Now transform to parallel axes, O$'x'$ and O$'y'$. From (5) and (6) of Art. 146, we have

$$y' = -\sqrt{2}, \qquad x' = \tfrac{3}{2}\sqrt{2},$$

which in (4) of Art. 146 gives

$$2y^2 - 4\sqrt{2}x = 0, \qquad \text{or} \qquad y^2 = 2\sqrt{2}x,$$

which is the equation of a parabola referred to its vertex and axis. [See Puckle's Conic Sections, p. 156.]

To construct the figure, let OX and OY represent the original axes; then, as $\theta = 45^\circ$, draw Ox, making with OX an angle of 45°, and draw Oy perpendicular to Ox; Ox and Oy will be the second set of axes. Locate the new origin O′ at $(\frac{3}{2}\sqrt{2},\ -\sqrt{2})$, as referred to the second set of axes, and draw the axes O′x' and O′y' parallel to Ox and Oy; O′x' will be the axis of the parabola, and O′y' will be tangent to it at the principal vertex. From the parameter, $2\sqrt{2}$, the curve may now be constructed as in Art. 52.

Remark 1.—The equation $y^2 = 2\sqrt{2}x$ might have been obtained immediately from (7) of Art. 146, by simply finding the values of c' and d'.

To find where the locus cuts the original axis of x, make $y = 0$ in (1), and get $x = 4 \pm 0$; that is, the curve is tangent to the axis of x at (4, 0).

Solving (1) for y, we get

$$y = x \pm \sqrt{8x - 16}. \tag{3}$$

For every value of $x < 2$, y is imaginary; when $x = 2$, $y = 2 \pm 0$, showing that the curve is tangent to the ordinate at the point (2, 2). For every value of $x > 2$, there are two values of y, one equal to that value of $x +$ the corresponding value of the radical, and the other equal to that value of $x -$ the corresponding value of the radical; that is, the line $y = x$ is such that if from any point of it whose abscissa > 2 we lay off a distance upward and also downward equal to the corresponding value of the radical, we shall determine two points of the curve; hence the curve is *symmetrical* with respect to the line $y = x$, which is therefore a diameter of the parabola, since it bisects a system of parallel chords. The equation of this diameter, $y = x$, shows that it passes through the origin, and is inclined to the axis of x at an angle of 45°, and hence it coincides with the axis Ox.

Remark 2.—In the above solution, we supposed 2θ to be in the *second quadrant*, where the tangent is *minus*, in which case when the

tangent reached its limit, $-\infty$, 2θ became 90°, and $\therefore$ $\theta = 45°$. We might have supposed 2θ to be in the *fourth quadrant*, estimated in the *negative* direction, where the tangent is minus, in which case when the tangent reached its limit, $-\infty$, 2θ would become $-90°$, and $\therefore$ $\theta = -45°$. If in this case we take the *positive* sign of the radical in (6) and (7) of Art. 143, we shall have $a' = 2$, $c' = 0$; and turning the axes through $-45°$, the final equation becomes $x^2 = 2\sqrt{2}y$, which is the equation of a parabola whose axis coincides with the axis of y; in this case, the final axis of x falls on $O'y''$, and the axis of y on $O'x'$.

4. Find the species and situation of

$$x^2 - 2xy + y^2 - 2c^2 = 0, \qquad (1)$$

and transform and construct.

$b^2 - 4ac = 0$; $\therefore$ the locus is a parabola. $\theta = -45°$.

From Art. 143, (6) and (7), we have $a' = 2$, $c' = 0$.

From Art. 146, (1), $d' = 0$, $e' = 0$, $f = -2c^2$.

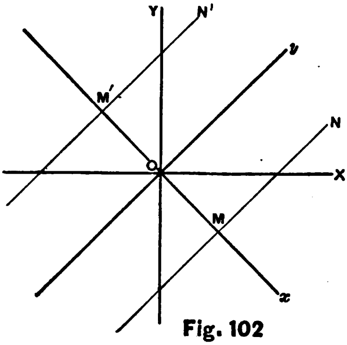

Fig. 102

Therefore (2) of Art. 146 becomes

$$2x^2 - 2c^2 = 0,$$

or $$x^2 - c^2 = 0,$$

or $$(x-c)(x+c) = 0, \qquad (2)$$

which represents two lines parallel to the axis of y, one c to the right, and the other c to the left of it.

This may be seen immediately by putting (1) in the form $(x - y - c\sqrt{2})(x - y + c\sqrt{2}) = 0$, which gives

$$y = x - c\sqrt{2} \quad \text{and} \quad y = x + c\sqrt{2}, \qquad (3)$$

which represents parallel lines making an angle of 45° with the axis of x.

To construct the figure, let OX and OY represent the original axes; then, as $\theta = -45°$, draw Ox making with OX an angle of $-45°$, and draw Oy perpendicular to Ox.

Ox and Oy will be the new axes. Now lay off OM and OM′ each $= c$, and draw MN and M′N′ parallel to Oy; they will be the required lines represented by (2) and also by (3), as is easily seen.

5. Find the species and situation of

$$5x^2 + 2xy + 5y^2 - 12c\sqrt{2}x - 12c\sqrt{2}y = 0, \qquad (1)$$

and transform.

$b^2 - 4ac < 0$; $\therefore$ the locus is an ellipse. $\theta = 45°$; by Art. 143, (6) and (7), we have $a' = 6$, $c' = 4$; by Art. 142, Sch., (6), (7), the centre is at $(c\sqrt{2},\ c\sqrt{2})$; by Art. 142, $f' = -24c^2$. Therefore, Art. 143, (5) becomes

$$6x^2 + 4y^2 - 24c^2 = 0,$$

or,
$$\frac{x^2}{4c^2} + \frac{y^2}{6c^2} = 1, \qquad (2)$$

which is the equation of an ellipse, the axis of x coinciding with the minor axis; the semi-axes are $c\sqrt{6}$ and $2c$.

6. Find the species and situation of

$$x^2 + 2xy - y^2 - 2cx + 2cy - 4c^2 = 0. \qquad (1)$$

$b^2 - 4ac > 0$; $\therefore$ an hyperbola. $\theta = 22\frac{1}{2}°$; $a' = \sqrt{2}$, $c' = -\sqrt{2}$ (Art. 143); centre at $(0, c)$ (Art. 142, Sch.); $f' = -3c^2$ (Art. 142); therefore, Art. 143, (5) becomes

$$x^2 - y^2 = \tfrac{3}{2}\sqrt{2}c^2,$$

which is an equilateral hyperbola.

7. Find the species and situation of

$$x^2 - 2xy + y^2 - 4c\sqrt{2}x = 0. \qquad (1)$$

$b^2 - 4ac = 0$; $\therefore$ a parabola. $\theta = -45°$; $a' = 2$, $c' = 0$ (Art. 143); $e' = -4c$ [Art. 146, (2)′].

Therefore [since c' vanishes instead of a' (see Remark of Art. 146)], (7) of Art. 146 becomes

$$a'x^2 + e'y = 0,$$

which in the present example becomes

$$2x^2 - 4cy = 0,$$

or

$$x^2 = 2cy,$$

which is the equation of a parabola whose axis coincides with the axis of y. (See Remark 2, Art. 146.)

8. Find the species and situation of the centre of the locus

$$3x^2 + 4xy + 5y^2 - 2x - 7y - 4 = 0,$$

and the inclination of its axis to the axis of x.

Ans. Ellipse; centre at $(-\frac{2}{11}, \frac{17}{22})$; $\theta = 58^\circ\ 17'$.

9. Find the species and situation of

$$x^2 + 2xy - y^2 + 8x + 4y - 8 = 0,$$

and transform to parallel axes through the centre.

Ans. $\begin{cases} \text{Hyperbola; } \theta = 22\frac{1}{2}^\circ\text{; centre, } (-3, -1); \\ \text{equation, } x^2 + 2xy - y^2 - 22 = 0. \end{cases}$

10. Find the species and situation of

$$14x^2 - 4xy + 11y^2 - 60 = 0,$$

and transform it to the axes of the curve.

Ans. $\begin{cases} \text{Ellipse; } \theta = \frac{1}{2}\tan^{-1}(-\frac{4}{3}); \\ \text{equation, } 3x^2 + 2y^2 - 12 = 0. \end{cases}$

11. Find the species and situation of

$$3x^2 + 8xy - 3y^2 + 6x - 10y + 5 = 0,$$

and transform to the axes of the curve.

Ans. $\begin{cases} \text{Hyperbola; } \theta = \frac{1}{2}\tan^{-1}\frac{4}{3}; \\ \text{centre, } (4.4, -1.08);\ 5x^2 - 5y^2 + 11.7 = 0; \\ \text{therefore, equilateral conjugate hyperbola.} \end{cases}$

12. Find the species and situation of

$$2x^2 + xy - 15y^2 - x + 19y - 6 = 0,$$

and transform to the axes of the curve.

Ans. Hyperbola; $\theta = \frac{1}{2}\tan^{-1}\frac{1}{17}$; centre, $(\frac{1}{11}, \frac{7}{11})$; equation is $2.01x^2 - 15.01y^2 = 0$; therefore (Art. 143), the locus is two intersecting right lines, which form a limiting case of the hyperbola.

13. Find the species and situation of

$$3x^2 - 8xy - 3y^2 + x + 17y - 10 = 0,$$

and transform to its centre.

Ans. Hyperbola; $\theta = \frac{1}{2}\tan^{-1}(-\frac{4}{3})$; centre, $(1.3, 1.1)$; equation $x^2 - y^2 = 0$; therefore the locus is an equilateral hyperbola in its limiting case; viz., two intersecting lines.

14. Find the species and situation of

$$x^2 - 4xy + 4y^2 - 2ax + 4ay = 0.$$

[The equation may be written

$$(x - 2y)(x - 2y - 2a) = 0,$$

and ∴ represents two parallel lines, which is one of the limiting cases of the parabola; the line parallel to them and midway between them is called a **Line of Centres.**] See Todhunter's Conic Sections, p. 240.

15. Find the species and centre of the locus

$$by\left(1 - \frac{y}{c}\right) + cx\left(1 - \frac{x}{b}\right) = xy.$$

Ans. Ellipse; centre at $\left(\frac{b}{3}, \frac{c}{3}\right)$.

16. Find the species and situation of $(y - x)^2 = ax$, and transform to the axis and vertex of the curve. (See Ex. 3.)

Ans. Parabola; $\theta = 45°$; equation is $y^2 = \frac{a}{4}\sqrt{2}x$.

17. In Fig. 101, find the lengths of OB and OO′, and the co-ordinates of O′ referred to the original axes; also prove that B is the extremity of the latus rectum of the parabola.

Ans. $OB = 2\sqrt{2}$; $OO' = \frac{1}{2}\sqrt{26}$; $ON = 2\frac{1}{2}$, $NO' = \frac{1}{2}$.

18. Find the species of $1 + 2x + 3y^2 = 0$, and transform to its axis and vertex. *Ans.* Parabola; $y^2 = -\frac{2}{3}x$.

19. Find the species of $3x^2 + 2y^2 - 2x + y - 1 = 0$, and transform to its axes. *Ans.* Ellipse; $72x^2 + 48y^2 = 35$.

20. Find the species of

$$x^2 - 10xy + y^2 + x + y + 1 = 0,$$

and transform to its axes.

Ans. Hyperbola; $32x^2 - 48y^2 = 9$.

21. Find the species of

$$x^2 - 2xy + y^2 - 6x - 6y + 9 = 0,$$

and transform to its axis and vertex.

Ans. Parabola; $y^2 = 3\sqrt{2}x$.

22. Show, by transformation, that

$$5x^2 - 4xy + y^2 - 4x + 2y + 2 = 0$$

represents an imaginary ellipse.

[The transformed equation is

$$(3 + 2\sqrt{2})x^2 + (3 - 2\sqrt{2})y^2 + 1 = 0;$$

∴ (Art. 143, Cor. 2), the locus is imaginary.]

23. Find the species and situation of the centre of

$$3x^2 + 4xy + y^2 - 5x - 6y - 3 = 0.$$

Ans. Hyperbola; centre at $(3\frac{1}{2}, -4)$.

24. Find the species and situation of

$$x^2 + 2xy - y^2 + 8x + 4y = 0.$$

Ans. Hyperbola; $\theta = 22\frac{1}{2}°$; centre, $(-3, -1)$.

25. Transform $2x^2 + 4xy + 3y^2 + 3x + y + \frac{5}{8} = 0$, to parallel axes through the centre of the curve.

Ans. $2x^2 + 4xy + 3y^2 - \frac{3}{2} = 0$; centre, $(-1\frac{3}{4}, 1)$.

26. Transform $2x^2 + 4xy + 3y^2 - 3 = 0$ to its axes.

Ans. $\frac{91}{60}x^2 + \frac{9}{60}y^2 = 1$, the axis of x coinciding with the *minor* axis of the ellipse. In this case we turned the old axes through $-\frac{1}{2}\tan^{-1}4$; had we turned them through $+\frac{1}{2}\tan^{-1}4$, and taken the *minus* value of the radical for a' in Art. 143, and *positive* value for c', we would have found, for the transformed equation, $\frac{9}{60}x^2 + \frac{91}{60}y^2 = 1$, the axis of x coinciding with the *major* axis of the curve. (See Remark, Ex. 3.)

CHAPTER IX.

HIGHER PLANE CURVES.

147. Higher Plane Curves are those whose equations are above the second degree, or which involve *transcendental* functions (Art. 17). It has been shown that every equation of the first degree between two variables represents a right line, and that every equation of the second degree between two variables represents a conic section; it follows that all other loci in a plane are *higher plane curves.*

An **Algebraic Curve** is one whose equation contains only algebraic functions. A **Transcendental Curve** is one whose equation contains transcendental functions. Many of the *higher plane curves* possess historical interest, from the labor bestowed on them by ancient mathematicians. We shall consider only a few of them.

THE CISSOID OF DIOCLES.

148. This curve was invented by Diocles, a Greek geometer who lived about the sixth century of the Christian era; the purpose of its invention was the solution of the problem of finding two mean proportionals. It may be defined as follows: If pairs of equal ordinates be drawn to the diameter of a circle, and through one extremity of this diameter and the point of intersection of one of the ordinates with the circumference a line be drawn, the locus of the intersection of this line and the equal ordinate, produced if necessary, is the **Cissoid of Diocles.**

The curve is constructed as follows: Let AB (Fig. 103) be the diameter of a circle; draw two equal ordinates MR and M'R'; join AR', cutting MR in P; then is P a point of the locus. In the same way, any number of points may be found. In like manner, draw through A and R a line cutting M′R′ produced in P′; P′ will be a point of the locus. In the same way, points can be found below AB.

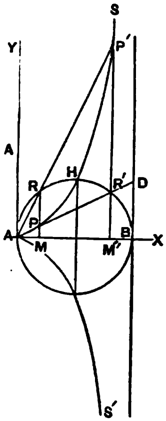

Fig. 103.

149. *To find the equation of the Cissoid of Diocles.*

I. *The rectangular equation.*

Let AX and AY be the axes; $AB = 2a$; and let (x, y) be any point P of the locus. Then we have

$$AM : PM :: AM' : R'M',$$

or
$$\frac{y}{x} = \frac{\sqrt{(2a - x)x}}{2a - x} = \frac{\sqrt{x}}{\sqrt{2a - x}};$$

Squaring and reducing, we have

$$y^2 = \frac{x^3}{2a - x}, \tag{1}$$

which is the required equation.

Sch.—Solving (1) for y, we have

$$y = \pm\sqrt{\frac{x^3}{2a - x}},$$

which shows that, for every value of $x < 2a$, y has two real values, numerically equal, with contrary signs; that is, the curve is symmetrical with respect to the axis of x. When $x = 2a$, $y = \infty$; hence the branches are infinite in length, and BD is an asymptote to them. When $x > 2a$, or negative, y is imaginary; therefore the locus is limited by $x = 0$ and $x = 2a$.

Sir Isaac Newton has given the following elegant construction of this curve by continuous motion: A right angle

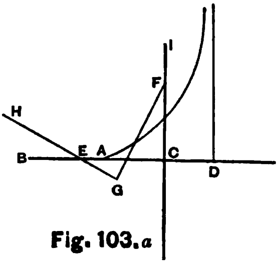

Fig. 103.*a*

has the side GF of fixed length, the point F moves along the fixed line CI, while the side GH passes through the fixed point E; a pencil at the middle point of GF will describe the Cissoid. The proof we leave to the student.

[See Salmon's Higher Plane Curves, p. 180.]

II. *The polar equation.*

Let A be the pole, and AB the initial line; let (r, θ) be any point P in the locus (see Fig. 103). Then, since $AM = BM'$, we have $AP = DR'$; therefore we have

$$\begin{aligned} r &= AD - AR' = AB \sec\theta - AB\cos\theta \\ &= 2a(\sec\theta - \cos\theta) \\ &= 2a\left(\frac{1-\cos^2\theta}{\cos\theta}\right) = 2a\frac{\sin^2\theta}{\cos\theta}; \end{aligned}$$

that is, $r = 2a\tan\theta\sin\theta$, which is the required equation.

SCH.—When $\theta = 0$, $r = 0$; when $\theta = 45°$, $r = a\sqrt{2}$; that is, H is a point in the curve. When $\theta = 90°$, $r = \infty$; when $\theta > 90°$ and $< 270°$, r is negative; while θ increases from 90° to 270°, the negative end of the radius-vector traces the branch AS' and the branch AS a second time; while θ increases from 270° to 360°, r is positive, and AS' is traced a second time; thus, the curve is traced twice by one revolution of the radius-vector.

THE CONCHOID OF NICOMEDES.*

150. This curve was invented by Nicomedes, who lived about the second century of our era, and was, like the preceding, first formed for the purpose of solving the problem of finding two mean proportionals, or the duplication of the cube; but it is more readily applicable to another problem not less celebrated among the ancients, that of the trisection of an angle. The curve may be defined as the locus of a point in a line which slides on and revolves about a fixed point, while the distance between the generating point and a fixed right line on either side of it is constant.

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* See Gregory's Examples, p. 130.

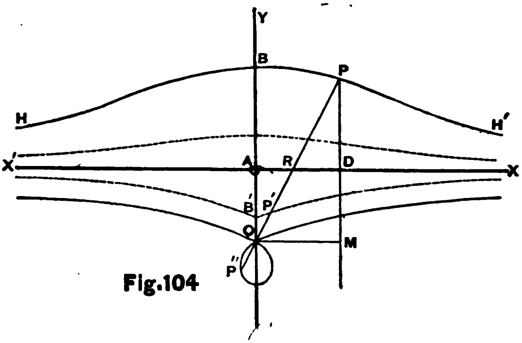

Fig.104

The curve is constructed as follows: Let O be the fixed point, XX′ the fixed right line, and AB the constant distance on the revolving line between the generating point and the fixed line. Draw through O any line, as OP; on OP, above XX′, lay off RP equal to AB; then will P be a point of the locus. In like manner, if we take AB′, below XX′, as a constant distance, and lay off RP′ equal to AB′, P′ will be a point of the locus.

151. *To find the equation of the Conchoid of Nicomedes.*

I. *The rectangular equation.*

Let XX′ and YY′ be the axes; $OA = p$; $AB = m$; and (x, y) any point P in the locus. Then we have, from the similar triangles PDR and PMO,

$$PD : DR :: PM : MO,$$

or, $$y : \sqrt{m^2 - y^2} :: y + p : x;$$
squaring and reducing, we have
$$x^2y^2 = (p + y)^2(m^2 - y^2), \tag{1}$$
which is the equation required.

SCH.—Solving (1) for x, we get
$$x = \pm \frac{p + y}{y} \sqrt{m^2 - y^2},$$
which shows that for every value of y, positive or negative, and numerically $< m$, x has two real values, numerically equal, with contrary signs; hence the curve has two branches, one above and one below the axis of x, both being symmetrical with respect to the axis of y. When y diminishes numerically, x increases and becomes ∞ when $y = 0$; hence the two branches are infinite in length, and the axis of x is an asymptote to them.

When $m > p$, for $y = -m$ or $-p$, $x = 0$; but for y *between* $-m$ and $-p$, x has two values, numerically equal, with contrary signs; hence the locus between these two limits is an oval symmetrical with respect to the axis of y. For y negative and less numerically than p, the values of x increase till they become $\pm \infty$ at $y = 0$.

When $m < p$, it is easily seen that there is no oval. The continuous line represents the case when $m > p$, and the broken line when $m < p$.

II. *The polar equation.*

Let O be the pole, OA the initial line, and (r, θ) any point P in the curve. Then we have
$$r = \text{OP} = \text{OR} + \text{RP} = \text{OA} \sec \theta + m;$$
that is, $r = p \sec \theta + m$, which is the required equation.

SCH.—When $\theta = 0$, $r = p + m$, and B is located; when $\theta = 90°$, $r = \infty$; when $\theta = 180°$, $r = -p + m$, and B′ is located; when $\theta > 90°$ and $< 270°$, $\sec \theta$ is negative, and the lower branch is traced by the negative end of the radius-vector; while θ increases from 270° to 360°, r is positive and the branch H′B is traced.

[For the application of this curve and also that of the Cissoid to the solution of the problem of finding two mean proportionals, see Gregory's Examples, p. 130.]

THE WITCH OF AGNESI.*

152. This curve was invented by Donna Maria Agnesi, an Italian lady, who lived in the eighteenth century. It may be defined as the locus of the extremity of an ordinate of a circle, produced till the produced ordinate is to the diameter of the circle as the ordinate itself is to one of the segments into which it divides the diameter.

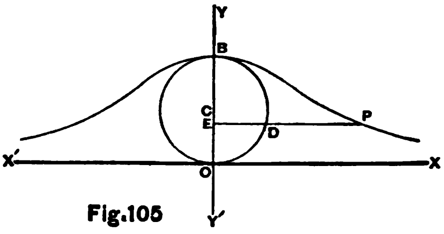

Fig. 105

To construct the Witch, let OB be the diameter of the circle; draw the ordinate ED; find the point P in ED produced so that

$$PE : OB :: ED : OE,$$

and P will be a point of the locus. In the same way, any number of points may be found.

153. *To find the equation of the Witch of Agnesi.*

Let XX′ and YY′ be the axes of co-ordinates, and (x, y) any point P in the locus. Call the diameter $2a$; then we have, from the definition,

$$x : 2a :: \sqrt{(2a-y)\,y} : y;$$

therefore, $x^2 y = 4a^2(2a-y)$, which is the required equation.

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* See Gregory's Examples, p. 131.

SCH.—When $y=0$, $x=\infty$; when $y=2a$, $x=0$; for every positive value of y between 0 and $2a$, x has two real values, numerically equal, with contrary signs, showing that the locus is symmetrical with respect to the axis of y, and is embraced between $y=0$ and $2a$, and has the axis of x for an asymptote.

THE LEMNISCATE OF BERNOUILLA.*

154. This curve was invented by James Bernouilla, who lived in the seventeenth century. It may be defined as the locus of the intersection of a tangent to an equilateral hyperbola with the perpendicular on it from the centre.

To find the equation of the Lemniscate.

I. *The rectangular equation.*

Let (x', y') be any point Q of the hyperbola at which the tangent is drawn; and let x and y be the current co-ordinates of the lines QP and OP. The equations of the hyperbola and the tangent are respectively

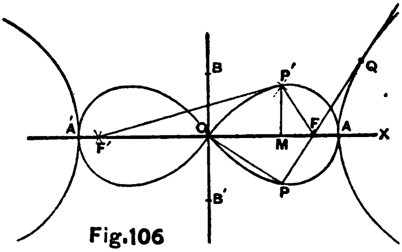

Fig.106

$$x'^2 - y'^2 = a^2, \tag{1}$$

and
$$xx' - yy' = a^2, \tag{2}$$

therefore the equation of OP is

$$y = -\frac{y'}{x'}x, \qquad \text{or} \qquad \frac{x}{x'} = -\frac{y}{y'}. \tag{3}$$

Multiplying (2) and (3) together, we get

$$x^2 + y^2 = \frac{a^2x}{x'} = -\frac{a^2y}{y'};$$

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* See Price's Calculus, Vol. I. p. 314.

therefore,

$$x' = \frac{a^2x}{x^2+y^2}, \quad \text{and} \quad y' = -\frac{a^2y}{x^2+y^2},$$

which in (1) gives,

$$\frac{a^4x^2}{(x^2+y^2)^2} - \frac{a^4y^2}{(x^2+y^2)^2} = a^2,$$

or, $$(x^2+y^2)^2 = a^2(x^2-y^2), \tag{4}$$

which is the required equation.

On transforming to polar co-ordinates, (4) becomes

$$r^4 = a^2r^2(\cos^2\theta - \sin^2\theta),$$

or, $$r^2 = a^2\cos 2\theta. \tag{5}$$

Sch.—When $\theta = 0$, $r = \pm a$; if we confine our attention to the positive values of r, we see that as θ increases from 0 to 45°, r diminishes from a to 0, and AP′O is traced; while θ increases from 45° to 135°, r is imaginary; when $\theta = 135°$, $r = 0$; while θ increases from 135° to 225°, r is real, and OA′O is traced; while θ increases from 225° to 315°, r is imaginary; while θ increases from 315° to 360°, r is real, and OPA is traced. The curve therefore consists of two ovals meeting at O; the tangents to the ovals at O coincide with the asymptotes of the equilateral hyperbola, and form angles of 45° with the axis of x (Art. 133, Sch.).

Sch. 2.—Take two points, F and F′, on opposite sides of O, at the distance $a\sqrt{\frac{1}{2}}$ from it, and take any point P′ in the curve; then we have

$$\text{FP}' = \sqrt{(a\sqrt{\tfrac{1}{2}} - x)^2 + y^2}, \tag{6}$$

and $$\text{F}'\text{P}' = \sqrt{(a\sqrt{\tfrac{1}{2}} + x)^2 + y^2}. \tag{7}$$

Multiply (6) and (7), and we have

$$\mathrm{FP}' \times \mathrm{F'P'} = \sqrt{(a\sqrt{\tfrac{1}{2}}-x)^2+y^2} \times \sqrt{(a\sqrt{\tfrac{1}{2}}+x)^2+y^2}$$

$$= \sqrt{(x^2+y^2)^2 - a^2(x^2-y^2) + \frac{a^4}{4}}$$

$$= \frac{a^2}{2}, \text{ by (4); that is,}$$

$$\mathrm{FP}' \times \mathrm{F'P'} = \frac{a^2}{2}.$$

Hence we may define the **Lemniscate** as a curve such that the product of the distances of any point in it from two fixed points, called the foci, is constant, and equal to the square of half the distance between the foci. (See Gregory's Examples, p. 132.)

[Let the student find the equation of the curve from this definition.]

We may construct the curve, from this latter definition, by points. Let F and F′ be the foci. With F as a centre, and any convenient radius, as FP′, describe an arc; with F′ as a centre, and a third proportional to FP′ and F′A, as F′P′, describe a second arc cutting the former at P′; then will P′ be a point in the locus. In the same way any number of points may be found.

THE CYCLOID.

155. The invention of this curve is usually ascribed to Galileo; it is generated by the motion of a point in the circumference of a circle which rolls along a fixed right line. Thus, if the circle NPB (Fig. 107) be rolled along the line OX, any point P in the circumference will describe a cycloid. The circle NPB is called the **Generating Circle** or **Generatrix**, and the point P the **Generating Point**. AX is called the **Base**, and is equal to the cir-

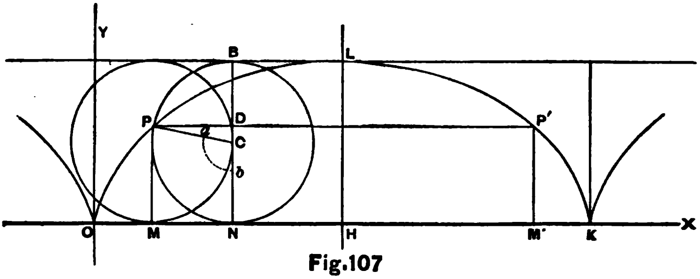

Fig.107

cumference of the generating circle. HL, perpendicular to the base at its centre, is the **Axis**, and is equal to the diameter of the generating circle, and L is the **Highest Point** of the cycloid.

156. *To find the equation of the cycloid referred to its base and a perpendicular at its left hand vertex.*

Let (x, y) be any point P in the cycloid OPLK, referred to the axes OX and OY; suppose that P has described the arc OP, while the generatrix has rolled from O to N, then ON = arc PN. Call the radius of the generatrix r. Then we have

$$x = \text{OM} = \text{ON} - \text{MN} = \text{arc PN} - \text{PD}$$

$$= r \text{ arc } ab - \sqrt{\text{ND} \times \text{DB}};$$

that is, $$x = r \operatorname{vers}^{-1}\frac{y}{r} - \sqrt{2ry - y^2}; \qquad (1)$$

which is the required equation of the cycloid, the arc ab being taken in the circle whose radius = 1.

SCH.—When y is negative, $\sqrt{2ry - y^2}$ is imaginary; therefore the curve lies only on the positive side of the base; when $y = 0$, $x = 0$, $2r\pi$, $4r\pi$, etc.; hence there is an infinite number of branches similar and equal to OLK, which is also evident from the mode of generation of the curve; when $y = 2r$, $x = \operatorname{vers}^{-1}, 2r = \pi r$, $3\pi r$, etc. For any one value of y, x has an infinite number of values, OM, OM′, etc.

It is frequently convenient to refer the cycloid to its *highest point* as origin, and to its *axis* as the axis of x.

157. *To find the equation of the cycloid referred to its highest point as its origin and to its axis as the axis of x.*

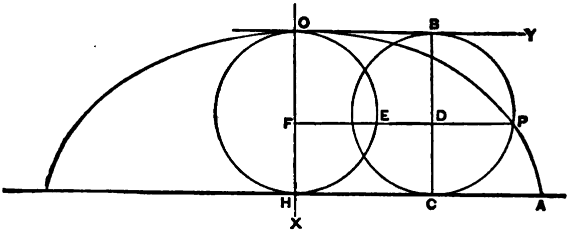

Fig. 108

Let (x, y) be any point P in the locus referred to the axes OX and OY; then we have

$$\begin{aligned} y &= \mathrm{PF} = \mathrm{PD} + \mathrm{DF} = \mathrm{PD} + \mathrm{CH} \\ &= \mathrm{PD} + \mathrm{AH} - \mathrm{AC} = \mathrm{PD} + \text{arc CPB} - \text{arc CP} \\ &= \mathrm{PD} + \text{arc PB} = \sqrt{\mathrm{CD} \times \mathrm{BD}} + \text{vers}^{-1}\, \mathrm{BD}\,; \end{aligned}$$

that is,

$$y = r\ \text{vers}^{-1}\frac{x}{r} + \sqrt{2ax - x^2} \quad \text{(see Art. 156)},$$

which is the required equation.

SCH.—When $x = 0$, $y = 0$; when $x = 2r$, $y = \text{vers}^{-1}\, 2r = \pi r, 3\pi r$, etc.; when x is negative, y is imaginary; for any one value of x, y has an infinite number of values.

After the conic sections there is no curve in geometry which has more exercised the ingenuity of mathematicians than the cycloid; and their labors have been rewarded by the discovery of a multitude of interesting properties, important both in geometry and in dynamics. [See Gregory's Examples, p. 136.]

SPIRALS.

158. We shall conclude this chapter with a brief account of *spirals,* many of which have been treated at length by old geometers. A **Spiral** is the locus of a point revolving about a fixed point, and constantly receding from it in accordance with some law. A right line then meets the curve in an infinity of points, and the curve is transcendental.

A **Spire** is the portion of the spiral generated in one revolution of the generating point.

The **Measuring Circle** is the circle whose radius is the radius-vector at the end of the first revolution of the generating point in the positive direction.

THE SPIRAL OF ARCHIMEDES.

159. This spiral was invented by Conon, but its principal properties were discovered by the geometer whose name it bears; it is the locus of a point revolving uniformly about a fixed point, and at the same time receding uniformly from it.

To construct the spiral of Archimedes.

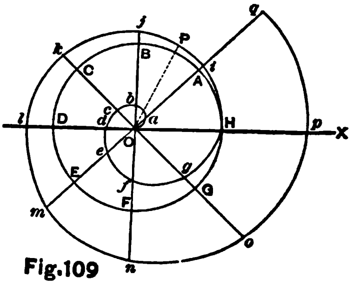

Fig. 109

Let O be the fixed point and OX the initial line; with O as a centre and any radius as OH, describe the circumference HADG; divide this circumference into any number of equal parts; for example, eight. On the radius OA lay off $Oa = \frac{1}{8}OH$; on OB lay off $Ob = \frac{2}{8}OH$; on OC lay off $Oc = \frac{3}{8}$ OH, etc.; the curve passing through these points, a, b, c, d, e, f, g, H, $i, \ldots p, q$, etc., will be the spiral of Archimedes, since the radius-vectors Oa, Ob, etc., increase uniformly, while the variable angle, estimated from OX, increases uniformly.

The circumference HADG is the measuring circle, O is the pole, O$abcdefg$H is the first spire, H$ijklmnop$ is the second spire, etc. The distance between any two consecutive spires measured on the radius-vector is equal at all points to OH, the radius of the measuring circle.

160. *To find the equation of the spiral of Archimedes.*

Let O be the pole (Fig. 109) and OX the initial line, and let (r, θ) be any point P in the spiral; then we have, from the definition, $r = a\theta$, as the required equation, when a is the ratio of r to θ.

Otherwise, we have from the figure,

$$\text{OP} : \text{OH} :: \theta : 2\pi;$$

or, calling the radius of the measuring circle a', we have

$$r : a' :: \theta : 2\pi;$$

therefore
$$r = \frac{a'\theta}{2\pi};$$

or writing a instead of $\frac{a'}{2\pi}$,

$$r = a\theta,$$

is the required equation.

When $\theta = 0$, $r = 0$; when $\theta = 2\pi$, $r = a'$; when $\theta = 4\pi$, $r = 2a'$; when $\theta = 6\pi$, $r = 3a'$, etc. The curve, therefore, starts at the pole, and the radius-vector, which is o at the beginning, becomes equal to OH $(= a')$, when it has made one revolution; and this is the distance between the points at which any radius-vector is cut by two successive spires.

THE RECIPROCAL OR HYPERBOLIC SPIRAL.

161. This spiral may be defined as the locus of a point revolving uniformly about a fixed point, and continually approaching it so that the radius-vector varies inversely as the variable angle.

To construct the Hyperbolic Spiral.

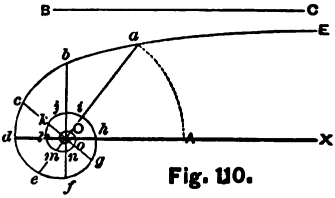

Fig. 110.

Let O be the pole and OX the initial line. Draw through O the lines Oa, Ob, Oc, etc., making equal angles with each other. Take a for a point of the spiral; lay off Ob = ½Oa; Oc = ⅓Oa, etc.; the curve passing through the points a, b, c, d, e, f, g, h, etc., will be the hyperbolic spiral, since the radius-vectors, Oa, Ob, etc., vary inversely as the variable angle estimated from OA.

The equation of the hyperbolic spiral follows directly from the definition, and is

$$r = \frac{a}{\theta}, \quad \text{or} \quad r\theta = a,$$

When $\theta \doteq 0$, $r = \infty$; that is, the curve approaches the initial line and touches it at infinity; when $\theta = 2\pi$, $r = \mathrm{O}h = a'$, which is the radius of the measuring circle; when $\theta = 4\pi$, $r = \frac{a'}{2}$, etc.; when $\theta = \infty$, $r = 0$; therefore, the curve continually approaches the pole as the radius-vector revolves, and reaches it after an infinite number of revolutions. From the equation $r = \frac{a}{\theta}$, it is evident that the arc Aa of the circle described with the radius Oa to any point of the curve, is constant and equal to a. [See Salmon's Higher Plane Curves, p. 280.]

THE LITUUS.

162. Another spiral worth mentioning is the **Lituus**, which may be defined as the locus of a point revolving uniformly about a fixed point, and continually approaching it so that the radius-vector varies inversely as the square root of the variable angle. Its equation therefore is

$$r = \frac{a}{\theta^{\frac{1}{2}}}.$$

SCH.—These spirals belong to one family, included in the general equation $r = a\theta^n$. When $n = 1$, we have $r = a\theta$, which is the spiral of Archimedes. When $n = -1$, we have $r = \frac{a}{\theta}$, which is the hyperbolic spiral. When $n = -\frac{1}{2}$, we have $r = \frac{a}{\theta^{\frac{1}{2}}}$, which is the Lituus.

THE CHORDEL.

162*a*. The **Chordel** is a plane curve, every point of which terminates an arc which originates in a fixed line, is described with a fixed point as a centre, and subtends a given length the same number of times as a chord.

The fixed line is called the **Directrix**, the fixed point the **Focus**, and the given length the **Element**.

A chordel in which the element is subtended n times as a chord, whose directrix is a right line, and whose focus is on the directrix, is called

A chordel of n elements, and rectilinear and focal directrix.[*]

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[*] This curve was invented by Mr. J. Bruen Miller; for an account of it see Van Nostrand's Engineering Magazine for March, 1880, pp. 206–209, which contains Mr. Miller's investigation of the chordel given geometrically, including the construction of the curve and its application to the division of an angle into n equal parts.

To find its equation.

Let the focus O be the pole, and the directrix OX be the initial line. Let (r, θ) be any point P of the curve, and

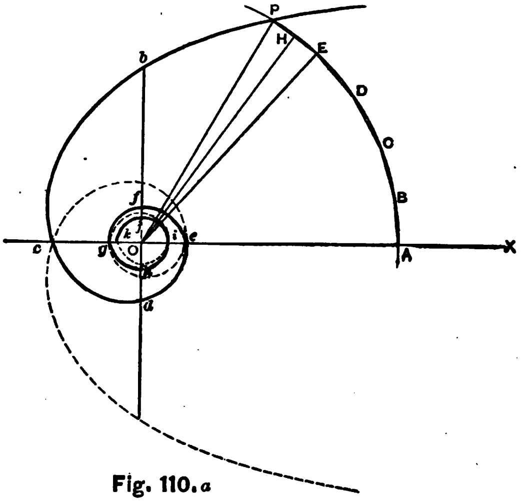

Fig. 110.*a*

$2a$ = an element AB = BC = etc. = EP, draw OH perpendicular to EP. Then we have

$$\sin \mathrm{HOP} = \frac{\mathrm{HP}}{\mathrm{OP}}; \quad \therefore \quad \sin\left(\frac{\theta}{2n}\right) = \frac{a}{r};$$

or
$$r = a \operatorname{cosec}\left(\frac{\theta}{2n}\right); \qquad (1)$$

which is the equation required.

Let $n = 5$ and $a = 1$; then (1) becomes $r = \operatorname{cosec}\left(\frac{\theta}{10}\right)$.

Letting $\theta = \frac{1}{2}\pi, \pi, \frac{3}{2}\pi, 2\pi, \frac{5}{2}\pi, 3\pi, \frac{7}{2}\pi, 4\pi, \frac{9}{2}\pi, 5\pi$, successively we get $r = 6.39, 3.24, 2.20, 1.70, 1.41, 1.24, 1.12, 1.05, 1.01$, and 1.00. Locating these values we have the points $b, c, d, e, f, g, h, i, j, k$; when $\theta = 0, r = \infty$. Now letting θ continue to increase, becoming $\frac{11}{2}\pi, 6\pi$, and so on to 10π, we get $r = 1.01, 1.05, 1.12$, and so on to infinity, the values being the same as those given above, when θ is increasing from 0 to 5π, except the order is inverted. Locating the second series of values we have the curve represented by the dotted line, which is the continuation of the part given in the full line, the two parts being symmetrical with respect to the line OX. While θ is increasing from 10π to 20π, r is negative and a second branch is traced by the negative end of the radius-vector, the two branches being symmetrically equal.

The essential merits of Mr. Miller's curve appear to be its mechanical construction, affording a mechanical multisection of any angle; and its very *general* definition, which will probably make the investigation of its properties rather fruitful. But such investigation would be out of place here.

THE LOGARITHMIC SPIRAL.

163. This spiral was invented by Descartes, and is the locus of a point so moving that the radius-vector increases in a *geometric*, while the variable angle increases in an *arithmetic* ratio. Its equation is therefore usually written

$$r = a^{\theta}.$$

To construct the Logarithmic Spiral.

Suppose $a = 2$, then $r = 2^{\theta}$; when $\theta = 0$, $r = 2^{0} = 1$, which gives the point a on the initial line OX, Fig. III. When $\theta = 1$, $r = 2^{1} = 2$; lay off the angle $XOb = 1 =$ arc of 57°.3, and take $Ob = 2$; b will be a point of the curve. When $\theta = 2$, $r = 2^{2} = 4$; lay off $XOc = 2 =$ arc of 114°.6, and take $Oc = 4$; c will be a point of the curve. The curve passing through a, b, c, etc., will be the logarithmic spiral.

When $\theta = -1 = XOb'$, $r = 2^{-1} = \frac{1}{2}$; lay off

$XOb' = -1 = -57°.3$, and take $Ob' = \frac{1}{2}$; b' will be a point of the curve. When $\theta = -2$, $r = 2^{-2} = \frac{1}{4}$; lay off $XOc' = -2 = -114°.6$, and take $Oc' = \frac{1}{4}$; c' will be a point of the curve; and so on for any number of points.

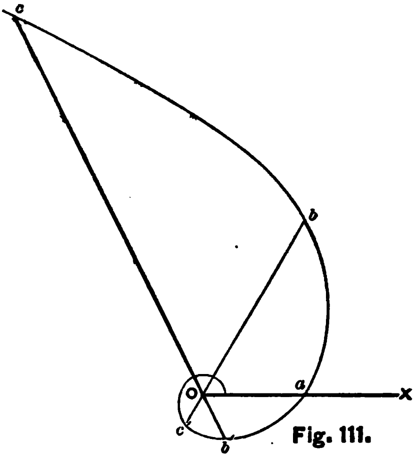

Fig. 111.

When $\theta = \infty$, $r = \infty$, hence the radius-vector will become infinite when it has made an infinite number of revolutions. When $\theta = -\infty$, $r = 0$; and therefore the spiral runs into its pole after an infinite number of revolutions in the negative direction.

The present chapter is but a brief sketch of Higher Plane Curves. The student who wishes to pursue the subject further, is referred to Salmon's Higher Plane Curves, Gregory's Examples, Price's Infinitesimal Calculus, and Cramer's Introduction to the Analysis of Curves.

PART II.

ANALYTIC GEOMETRY OF THREE DIMENSIONS.

CHAPTER I.

THE POINT.

164. We have seen (Art. 5) that the position of a point in a plane is determined by referring it to two co-ordinate axes, OX, OY, drawn in the plane. We shall now show that the position of a point in *space* may be determined by referring it to three *co-ordinate planes*.

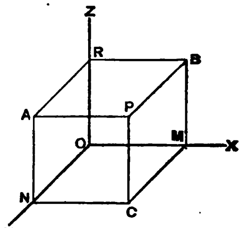

Fig. 111. *a*

Let XOY, YOZ, ZOX be three planes of indefinite extent, intersecting each other in the three right lines OX, OY, OZ. Now, if through any point P in the surrounding space we draw PA parallel to OX, PB parallel to OY, and PC parallel to OZ, it is plain that the position of P with reference to the three planes is known, if the lengths of PA, PB, and PC are known. For example, if we have given PA $= a$, PB $= b$, PC $= c$, we can determine the position of the point P with reference to the three planes as follows: We measure OM ($= a$) along OX, and ON ($= b$) along OY, and draw the parallels MC and NC; then at the intersection C measure

CP ($= c$) on a line parallel to OZ; P will be the point whose position we wished to determine. Otherwise thus: having measured OM, ON, OR, equal respectively to a, b, c, pass through M the plane PCBM parallel to the plane yz; through N the plane PACN, parallel to zx; and through R the plane PABR, parallel to xy; the intersection of the three planes so drawn is the point P.

165. The three planes XOY, YOZ, ZOX, are called the **Co-ordinate Planes**, and are designated as the planes xy, yz, zx, respectively. The three lines OX, OY, OZ, in which these planes intersect, are called the **Co-ordinate Axes**; OX is called the axis of x, OY the axis of y, and OZ the axis of z. The point O in which the three axes intersect, and which is therefore common to the three co-ordinate planes, is called the **Origin**. The distances PA, PB, PC, or their equals, OM, ON, OR, are called the **Rectilinear Co-ordinates** of P, and are respectively represented by x, y, z. The co-ordinate axes may be inclined to each other at any angle whatever; and they are said to be rectangular or oblique, according as the angles at which they intersect are right or oblique angles. In this work we shall employ rectangular axes, as they are the most simple, and can always be secured by a proper transformation.

The co-ordinates of a point are the distances of the point from the three co-ordinate planes yz, zx, xy; hence, if the co-ordinates of a point are respectively denoted by a, b, c, we have for the point,

$$x = a, \qquad y = b, \qquad z = c,$$

which are the **Equations of the Point**. When these equations are given, the point is said to be given, and may be constructed as in Art. 164; the point whose position is defined by the above equation is commonly spoken of as the point (a, b, c).

166. The plane xy is supposed to be horizontal, as the plane of the floor on which the student is standing; the plane xz is perpendicular to the first, and in front of the student; the plane yz is perpendicular to both the others, and on the left of the student.

These co-ordinate planes divide the surrounding space into eight solid angles, which are numbered as follows: The **First** angle lies *above* the plane xy, in *front* of the plane xz, and to the *right* of the plane yz; the **Second** is to the left of the first; the **Third** is behind the second; the **Fourth** is behind the first; the **Fifth, Sixth, Seventh** and **Eighth** are below the first, second, third, and fourth, respectively.

167. In order that the equations $x = a$, $y = b$, $z = c$ should be satisfied by only one point, it is necessary to consider not only the absolute values of the co-ordinates, but also their signs. It is customary to consider lines measured upwards as positive, and hence those measured downwards must be considered negative; also those measured towards the right are considered positive, and hence those measured towards the left are negative; also those measured towards the front are considered positive, and hence those measured towards the rear are negative. Hence, by giving the co-ordinates their proper signs, we may represent a point in either of the eight angles by one of the following sets of equations:

$$\text{First Angle,}\quad \left\{\begin{aligned} x &= +a,\\ y &= +b,\\ z &= +c, \end{aligned}\right\}\quad \text{or by}\quad (a,\ b,\ c).$$

$$\text{Second “}\quad \left\{\begin{aligned} x &= -a,\\ y &= +b,\\ z &= +c, \end{aligned}\right\}\quad \text{“}\quad (-a,\ b,\ c).$$

$$\text{Third “}\quad \left\{\begin{aligned} x &= -a,\\ y &= -b,\\ z &= +c, \end{aligned}\right\}\quad \text{“}\quad (-a,\ -b,\ c).$$

Fourth Angle, $\left\{\begin{array}{l} x = +a, \\ y = -b, \\ z = +c, \end{array}\right\}$ or by $(a, -b, c)$.

Fifth " $\left\{\begin{array}{l} x = +a, \\ y = +b, \\ z = -c, \end{array}\right\}$ " $(a, b, -c)$.

Sixth " $\left\{\begin{array}{l} x = -a, \\ y = +b, \\ z = -c, \end{array}\right\}$ " $(-a, b, -c)$.

Seventh " $\left\{\begin{array}{l} x = -a, \\ y = -b, \\ z = -c, \end{array}\right\}$ " $(-a, -b, -c)$.

Eighth " $\left\{\begin{array}{l} x = +a, \\ y = -b, \\ z = -c, \end{array}\right\}$ " $(a, -b, -c)$.

COR.—Any point in the plane xy evidently has its $z = 0$; hence, equations of a point in this plane are $x = a$, $y = b$, $z = 0$, or the point is $(a, b, 0)$; and there are similar equations for points in each of the other co-ordinate planes.

Any point on the axis of x has its y and z each $= 0$; hence the equations of a point on this axis are $x = a$, $y = 0$, $z = 0$, or the point is $(a, 0, 0)$; and there are similar equations for points on each of the other co-ordinate axes.

At the origin we evidently have $x = 0$, $y = 0$, $z = 0$, which are the three equations of *the origin of co-ordinates.*

168. The **Orthogonal Projection** of a point on a line or a plane is the foot of a perpendicular from the point to the line or plane. In the present work, when we use the term *projection,* we shall always mean an *orthogonal* projection, since the axes are rectangular. The points M, N, R, are the projections of the point P on the three co-ordinate axes, and the points A, B, C, are the projections of the point P on the three co-ordinate planes.

The projection of a *given right line* upon another right line, or upon a plane, is the line which joins the projections of the extremities of the given line. Thus, OM, ON, OR, are the projections of OP upon the co-ordinate axes x, y, and z respectively; and the lines OA, OB, OC are the projections of OP upon the co-ordinate planes yz, zx, xy, respectively.

The angle which any right line makes with a *plane* is the angle which the line makes with its projection on that plane; the angle which it makes with a given *line* is the angle which it makes with a line drawn through any point of it and parallel to the given line.

It is clear that the projection of a finite right line upon another right line or upon a plane is equal to the first line multiplied by the cosine of the angle which it makes with the second line or with the plane. Hence, it is also evident that the projections of any area of a plane upon another plane is equal to the original area multiplied by the cosine of the angle between the planes.

The line that determines the projection of a point upon a line or plane is called the **Projecting Line** of that point. The projection of any *curve* upon a plane is the curve formed by projecting all of its points. The projecting lines of the different points form a *cylindrical surface* which is called the **Projecting Cylinder** of the curve. When the curve projected is a right line, the projecting cylinder becomes a plane called the **Projecting Plane** of the line.

169. *To find a formula for the distance between two points in space whose co-ordinates are known.*

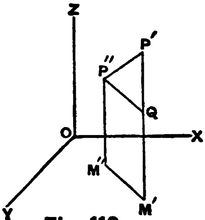

Fig. 112.

Let (x', y', z') and (x'', y'', z'') be the two points P′ and P″. Let the projection of P′P″ on the plane xy be M′M″; draw P″Q parallel to M″M′; represent the distance P′P″ by d. Then we have

$$\overline{P'P''}^2 = \overline{P'Q}^2 + \overline{QP''}^2.$$

But $\quad P'Q = z' - z''$;

and $\quad \overline{QP''}^2 = \overline{M'M''}^2$

$$= (x' - x'')^2 + (y' - y'')^2. \quad \text{(Art. 9.)}$$

Therefore, $\quad d^2 = (x'-x'')^2 + (y'-y'')^2 + (z'-z'')^2,$

or $$d = \sqrt{(x'-x'')^2+(y'-y'')^2+(z'-z'')^2}. \qquad (1)$$

The quantities $(x' - x'')$, $(y' - y'')$, $(z' - z'')$ are equal to the projections of d on the co-ordinate axes x, y, and z, respectively. Hence, *the square of the length of any right line in space is equal to the sum of the squares of its projections on any three rectangular axes.*

COR.—If one of the points, as P″, were the origin, we would have, from (1),

$$d = \sqrt{x'^2 + y'^2 + z'^2}, \qquad (2)$$

which is the distance of any point (x', y', z') from the origin. Hence, *the square of the radius-vector of any point is equal to the sum of the squares of the co-ordinates of the point.*

170. The position of a point is sometimes expressed by its radius-vector and its **Direction Cosines**; that is, the cosines of the three angles which the radius-vector makes with the three co-ordinate axes (see Art. 22, III, Sch. 1); the angles themselves are called the **Direction Angles.** Let these three angles be α, β, γ; then, since the co-ordinates x, y, z of the point are the projections of its radius-vector on the three axes (Art. 168), we have

$$x = \rho \cos \alpha, \qquad y = \rho \cos \beta, \qquad z = \rho \cos \gamma. \qquad (1)$$

Squaring and adding these equations, and remembering that $\rho^2 = x^2 + y^2 + z^2$ (Art. 169, Cor.), we get

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1, \qquad (2)$$

which expresses the relation between the direction-cosines of the radius-vector. That is, *the sum of the squares of the direction-cosines of any line is* = 1.

171. The position of any point P (Fig. 111), may be expressed by its *polar co-ordinates;* viz., the *radius-vector* OP ($=\rho$); the angle POR ($=\gamma$), which the radius-vector makes with the axis of z; and the angle XOC ($=\phi$), which OC, the projection of the radius-vector on the plane xy, makes with the axis of x. These angles are called the **Vectorial Angles**, and O is called the **Pole.**

From the figure we have

$$x = \rho \sin \gamma \cos \phi,$$

$$y = \rho \sin \gamma \sin \phi,$$

$$z = \rho \cos \gamma,$$

which are the formulæ for transforming from rectangular to polar co-ordinates.

CHAPTER II.

THE RIGHT LINE.

172. *To find the equations of a right line in space.*

Since a line in space is known when two of its *projections* are known (see Church's Desc. Geom., Art. 12), we need only find the equations of the *projections* of the line upon two of the co-ordinate planes.

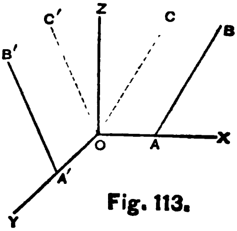

Fig. 113.

Let AB and A′B′ be the projections of a right line on the co-ordinate planes xz and yz. Draw through the origin OC and OC′, parallel respectively to AB and A′B′. Let (x, z) be any point in AB, and (y, z) be any point in A′B′; let $a =$ tangent of COZ, and $b =$ tangent of C′OZ; and let α and β be the intercepts OA and OA′ respectively. Then we have

$$x = az + \alpha, \tag{1}$$

and

$$y = bz + \beta, \tag{2}$$

for the equations of the projections of a right line on the co-ordinate planes xz and yz.

Now, since the x and z of *any point in the given line in space* are equal and parallel to the x and z of the *projection* of the same point on the plane xz, it follows that (1) expresses the relation between the x and z of every point of the given line. Also, since the y and z of *any point in the given line in space* are equal and parallel to the y and z of the *projection* of the same point on the plane yz, it follows that (2) expresses the relation between the y and z of every point of the given line. Hence, making (1) and (2) *simultaneous,* that is, making the co-ordinates the same in both equations, they together will express the relation between the co-ordinates x, y, z of every point of the given line; *therefore* (1) *and* (2) *are the equations required.*

Cor. 1.—Combining (1) and (2), and eliminating z, we obtain

$$y - \beta = \frac{b}{a}(x - \alpha), \tag{3}$$

which expresses the relation between the x and y of every point of the given line; hence it is the equation of the projection of the line on the plane xy.

Cor. 2.—If $y = 0$, we get

$$z = -\frac{\beta}{b}, \qquad x = \frac{b\alpha - a\beta}{b};$$

hence the line pierces the plane xz in the point

$$\left(\frac{b\alpha - a\beta}{b},\ 0,\ -\frac{\beta}{b}\right).$$

Similarly, we find it pierces the plane yz in the point

$$\left(0,\ \frac{a\beta - b\alpha}{a},\ -\frac{\alpha}{a}\right),$$

and the plane xy in the point

$$(\alpha,\ \beta,\ 0).$$

Cor. 3.—If the line passes through the origin, we have α and β equal to 0; therefore (1) and (2) become

$$x = az, \qquad y = bz, \tag{4}$$

which are *the equations of a line in space passing through the origin.*

173. *To find the equations of a right line in space,*

I. *Passing through a given point;*

II. *Passing through two given points; and*

III. *Passing through a given point, and making the angles α, β, γ with the co-ordinate axes.*

I. Let (x', y', z') be a given point, and let the equations of the right line be

$$x = az + \alpha, \tag{1}$$

$$y = bz + \beta. \tag{2}$$

Since the point (x', y', z') is to be on the line, it must satisfy its equations, giving us

$$x' = az' + \alpha, \tag{3}$$

$$y' = bz' + \beta. \tag{4}$$

Eliminating α and β by subtracting (3) from (1), and (4) from (2), we get

$$x - x' = a(z - z'), \tag{5}$$

$$y - y' = b(z - z'), \tag{6}$$

for *the equations of a right line passing through a given point in space.*

II. Let (x'', y'', z'') be the second given point. Since this point is to be on the line, it must satisfy its equations, giving us

$$x'' = az'' + \alpha, \tag{7}$$

$$y'' = bz'' + \beta. \tag{8}$$

Eliminating α and β by subtracting (7) from (3) and (8) from (4), we get

$$x' - x'' = a(z' - z''), \quad \text{or} \quad a = \frac{x' - x''}{z' - z''}; \tag{9}$$

$$y' - y'' = b(z' - z''), \quad \text{or} \quad b = \frac{y' - y''}{z' - z''}. \tag{10}$$

Substituting these values of a and b in (5) and (6), we get

$$x - x' = \frac{x' - x''}{z' - z''}(z - z'), \tag{11}$$

$$y - y' = \frac{y' - y''}{z' - z''}(z - z'), \tag{12}$$

which are *the equations of a right line passing through two given points in space;* or, as they may be more symmetrically written,

$$\frac{x - x'}{x' - x''} = \frac{y - y'}{y' - y''} = \frac{z - z'}{z' - z''}.$$

III. Let (x, y, z) be any variable point on the line. By Art. 169, $x - x'$, $y - y'$, $z - z'$ are the projections of the distance between the points (x', y', z') and (x, y, z) on the axes; and since this distance is equal to its projection on either of the axes divided by the corresponding direction-cosine, we have

$$\frac{x - x'}{\cos \alpha} = \frac{y - y'}{\cos \beta} = \frac{z - z'}{\cos \gamma}, \tag{13}$$

which are the equations required, and are known as the *symmetrical equations of a right line in space.* (See Art. 22, II).

174. *To find the angle between two right lines in space in terms of the angles which they make with the co-ordinate axes.*

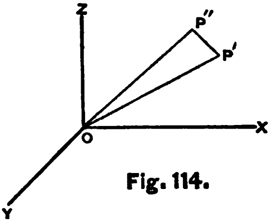

Fig. 114.

The angle between any two right lines in space is equal to the angle between two lines drawn through any given point, and parallel respectively to the given lines. Therefore, let OP′ and OP″ be drawn through the origin and parallel to the given lines; the angle between OP′ and OP″ will be equal to the required angle.

Let (x', y', z') and (x'', y'', z'') be the points P′ and P″ respectively, and $OP' = r'$, $OP'' = r''$, $P'P'' = d$; also, let the angles which OP′ and OP″ make with the co-ordinate axes be α, β, γ, and α', β', γ', respectively; and denote the angle P′OP″ by v.

Then, by Trigonometry, we have

$$\cos v = \frac{r'^2 + r''^2 - d^2}{2r'r''}. \tag{1}$$

But (Art. 169) we have

$$d^2 = (x' - x'')^2 + (y' - y'')^2 + (z' - z'')^2. \tag{2}$$

Substituting (2) in (1), and remembering that

$$x'^2 + y'^2 + z'^2 = r'^2, \qquad x''^2 + y''^2 + z''^2 = r''^2,$$

we get

$$\cos v = \frac{x'x'' + y'y'' + z'z''}{r'r''}. \tag{3}$$

But (Art. 170) we have

$$x' = r' \cos \alpha, \quad y' = r' \cos \beta, \quad z' = r' \cos \gamma; \tag{4}$$

$$x'' = r'' \cos \alpha', \quad y'' = r'' \cos \beta', \quad z'' = r'' \cos \gamma', \tag{5}$$

which in (3) give

$$\cos v = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'. \tag{6}$$

That is, *the cosine of the angle between two right lines in space is equal to the sum of the products of the cosines of the angles formed by these lines with the co-ordinate axes.*

175. *To find the angle between two right lines in space in terms of the tangents of the angles which the projections of the lines on the planes xz and yz make with the axis of z.*

The equations of OP′ and OP″ (Art. 172, Cor. 3) are,

(OP′), $\quad x = az, \quad y = bz,$ (1)

and (OP″), $\quad x = a'z, \quad y = b'z.$ (2)

Since (x', y', z') is on OP′, it must satisfy (1), giving us

$$x' = az'. \qquad y' = bz'. \tag{3}$$

Since (x'', y'', z'') is on OP″, it must satisfy (2), giving

$$x'' = a'z'', \qquad y'' = b'z''. \tag{4}$$

Substituting these values of x' and y' given in (3) in

$$x'^2 + y'^2 + z'^2 = r'^2,$$

we get $$a^2z'^2 + b^2z'^2 + z'^2 = r'^2,$$

or $$z' = \frac{r'}{\sqrt{a^2 + b^2 + 1}};$$

which in (3) gives us

$$x' = \frac{ar'}{\sqrt{a^2 + b^2 + 1}},$$

$$y' = \frac{br'}{\sqrt{a^2 + b^2 + 1}}.$$

Now these values of x', y', z' in (4) of Art. 174 give us

$$\cos\alpha = \frac{a}{\sqrt{a^2 + b^2 + 1}}, \tag{5}$$

$$\cos\beta = \frac{b}{\sqrt{a^2 + b^2 + 1}}, \tag{6}$$

$$\cos\gamma = \frac{1}{\sqrt{a^2 + b^2 + 1}}. \tag{7}$$

In like manner, we find

$$\cos\alpha' = \frac{a'}{\sqrt{a'^2 + b'^2 + 1}},$$

$$\cos\beta' = \frac{b'}{\sqrt{a'^2 + b'^2 + 1}},$$

$$\cos\gamma' = \frac{1}{\sqrt{a'^2 + b'^2 + 1}}.$$

Substituting these values of the cosines in (6) of Art. 174, we get

$$\cos v = \frac{aa' + bb' + 1}{\sqrt{a^2 + b^2 + 1}\sqrt{a'^2 + b'^2 + 1}}. \tag{8}$$

Cor. 1.—If the lines are parallel to each other, $v = 0$, and $\cos v = 1$; hence, clearing (8) of fractions and squaring, it becomes

$$(a^2 + b^2 + 1)(a'^2 + b'^2 + 1) = (aa' + bb' + 1)^2;$$

transposing and uniting, we obtain

$$(a - a')^2 + (b - b')^2 + (ab' - a'b)^2 = 0.$$

Each term being a square, and therefore positive, this equation can be satisfied only when the terms are separately equal to 0, giving us

$$a = a', \qquad b = b', \qquad ab' = a'b.$$

But the third term follows directly from the other two; hence,

$$a = a' \quad \text{and} \quad b = b' \tag{9}$$

are *the equations of condition* that two lines in space shall be parallel to each other; that is, *if two right lines in space are parallel, their projections on the co-ordinate planes are parallel.* [Art. 172, Eqs. (1), (2), (3); also Art. 27, Cor. 1.]

Cor. 2.—If the lines are perpendicular to each other, $\cos v = 0$, and hence (8) becomes

$$aa' + bb' + 1 = 0, \tag{10}$$

which is *the equation of condition* that makes two right lines in space perpendicular to each other.

176. *To find the condition that two right lines in space may intersect, and the position of the point of intersection.*

Let
$$\begin{cases} x = az + \alpha, & (1) \\ y = bz + \beta, & (2) \end{cases}$$

and
$$\begin{cases} x = a'z + \alpha', & (3)\\ y = b'z + \beta', & (4)\end{cases}$$

be the equations of two right lines in space which intersect. If these lines do intersect, the co-ordinates of the point of intersection must satisfy all the equations. But as there are *four* equations, containing only *three* unknown quantities, the equations cannot all be satisfied by the same set of values of x, y, z, if they are independent of each other. That is, there must be such a relation between the known quantities as to make one equation depend upon the other three; and the equation expressing this relation will be the required condition of intersection.

We form this condition, of course, by eliminating x, y, z from the four equations. Solving (1) and (3), and also (2) and (4) for z, we get

$$z = \frac{\alpha' - \alpha}{a - a'}, \tag{5}$$

and
$$z = \frac{\beta' - \beta}{b - b'}. \tag{6}$$

Equating the two values of z in (5) and (6), we get

$$\frac{\alpha' - \alpha}{a - a'} = \frac{\beta' - \beta}{b - b'}; \tag{7}$$

which is *the required condition that two lines in space shall intersect.*

Substituting (5) in (1), and (6) in (2), we get

$$x = \frac{a\alpha' - a'\alpha}{a - a'};$$

$$y = \frac{b\beta' - b'\beta}{b - b'}.$$

These values of x and y, with the value of z from either (5) or (6), will give *the point of intersection* when (7) is satisfied.

EXAMPLES.

1. Find the distance between the points (3, 2, 1) and (4, 5, 3). *Ans.* $d = \sqrt{14}$.

2. Find the distance between the points (−5, 5, −3) and (1, 0, 5). *Ans.* $d = 11.18$.

3. Find the equations of a right line passing through the point (2, 3, 4).

Ans. $x - 2 = a(z - 4)$; $y - 3 = b(z - 4)$.

4. Find the equations of the right line passing through the two points (3, 4, 2) and (4, 1, 5).

Ans. $3x = z + 7$; $3y = -3z + 18$.

5. Find the points in which the line last found pierces the co-ordinate planes.

Ans. $(2\frac{1}{3}, 6, 0)$, $(4\frac{1}{3}, 0, 6)$, and $(0, 13, -7)$.

6. Find the equation of the projection of the line in Ex. 4, on the plane xy. *Ans.* $3x = -y + 13$.

7. The equations of the projections of a right line on zx, yz, are

$$x = z + 1, \qquad y = \tfrac{1}{2}z - 2;$$

required its equation on the plane xy.

Ans. $2y = x - 5$.

8. Find the equations of the three projections of a right line which passes through the two points (2, 1, 0) and (− 3, 0, − 1).

Ans. $x = 5z + 2$; $y = z + 1$; $5y = x + 3$.

9. Find the angle between the right lines

$$x = 3z + 5, \qquad y = 5z + 3;$$

and

$$x = z + 1, \qquad y = 2z.$$

Ans. 14° 58′.

10. Find the equations of a right line through the origin and perpendicular to both the lines in Ex. 9.

Ans. $x = 3z$; $y = -2z$.

11. Find the cosine of the angle between the lines

$$x = 2z + 1, \qquad y = 2z + 2;$$

and $$x = z + 5, \qquad y = 4z + 1.$$

Ans. $\cos v = \dfrac{11}{9\sqrt{2}}$.

12. Find the point of intersection of the two lines

$$x = -2z + 3, \qquad y = z - 2;$$

and $$x = 3z - 1, \qquad 5y = -10z + 2;$$

and the cosine of the angle between them.

Ans. $(\frac{7}{5}, -\frac{6}{5}, \frac{4}{5})$, $\cos v = \pm\sqrt{\frac{7}{12}}$.

13. Find whether the two lines

$$x = 2z + 1, \qquad y = 3z + 4;$$

and $$x = -2z + 3, \quad y = z - 2;$$

are parallel or perpendicular to each other.

Ans. Perpendicular.

14. Find the equations of the line which passes through the point $(-3, 2, -1)$ and is parallel to the line

$$x = -3z - 1, \qquad y = 4z + 3;$$

(see Art. 175, Cor. 1), also of the line through the same point and perpendicular to the same line. (See Art. 175, Cor. 2, and Art. 176.)

Ans. To first, $x = -3z - 6, \quad y = 4z + 6;$

" To second, $27x = 49z - 32, \quad 9y = 10z + 28.$

15. Find the direction-cosines of

$$x = 4z + 3, \qquad y = 3z - 2.$$

Ans. $\cos\alpha = \dfrac{4}{\sqrt{26}}$; $\cos\beta = \dfrac{3}{\sqrt{26}}$; $\cos\gamma = \dfrac{1}{\sqrt{26}}$.

16. Find the equation of a right line through the point $(4, 5, 7)$, its direction-cosines being $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$.

Ans. $\dfrac{x-4}{2} = \dfrac{y-5}{1} = \dfrac{z-7}{2}$; or $\left\{\begin{matrix} x = z - 3 \\ 2y = z + 3 \end{matrix}\right\}$.

17. A right line makes an angle of 60° with one axis and 45° with another. What angle does it make with the third axis? (Art. 170.) *Ans.* 60°.

18. Find the angles which the line $x = -2z + 1$, $y = z + 3$, makes with the co-ordinate axes.
Ans. $\alpha = 144° \, 44'$; $\beta = 65° \, 54'$; $\gamma = 65° \, 54'$. (Art. 175.)

19. The equations of two lines are

$$x = 2z + 1, \qquad y = 2z + 2;$$

and

$$x = z + 5, \qquad y = 4z + \beta';$$

find the value of β' so that the lines shall intersect each other, and also the point of intersection. (Art. 176.)
Ans. $\beta' = -6$; the point of intersection is (9, 10, 4).

20. Find the angle between the lines

$$x = z\sqrt{2}, \qquad y = z\sqrt{\tfrac{3}{2}};$$

and

$$x = y\sqrt{3}, \qquad z = 0.$$

[Here $b' = \infty$ and $a' = \infty\sqrt{3}$. See Art. 172.]
Ans. 30°.

CHAPTER III.

THE PLANE.

177. The Equation of a Plane is the equation which expresses the relation between the co-ordinates of every point of the plane.

To find the equation of a plane.

A plane may be generated by revolving a right line about its intersection with another right line, to which it is perpendicular. The revolving line is called the **Generator**, and the line to which it is perpendicular is called the **Director.**[*]

Let $$x = az + \alpha, \qquad y = bz + \beta, \tag{1}$$

be the equation of a given line which we take for the director. If the director passes through the point (x', y', z') its equations will be

$$x - x' = a(z - z'); \qquad y - y' = b(z - z'). \tag{2}$$

The equations of a line through the same point (x', y', z') and perpendicular to the director are

$$x - x' = a'(z - z'); \qquad y - y' = b'(z - z'). \tag{3}$$

The equation of condition that makes (3) perpendicular to (2) is (Art. 175, Cor. 2)

$$aa' + bb' + 1 = 0. \tag{4}$$

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[*] In using these words I follow Gregory and Salmon, instead of giving them a feminine termination, and calling them "*generatrix*" and "*directrix*." Also, the word "directrix" has already been used in a different sense (see Art. 51) from the present, and it is well to distinguish between the two.

Substituting in (4) the values of a' and b' found in (3), we have

$$a\frac{x-x'}{z-z'} + b\frac{y-y'}{z-z'} + 1 = 0;$$

or

$$ax + by + z - (z' + ax' + by') = 0. \qquad (5)$$

Now for only *one set* of values of a' and b' in (3) that will satisfy (4), x, y, z, in (5) are the co-ordinates of every point of the generator in one position of it; likewise for a *second set* of values of a' and b' in (3) that would satisfy (4), x, y, z, in (5) are the co-ordinates of every point of the generator in this second position. Therefore for *every set* of values of a' and b' in (3) that will satisfy (4), x, y, z, in (5) are the co-ordinates of every point of the generator in every position of it; that is, they are the co-ordinates of the plane. Hence, (5) is the equation of the plane.

Representing the constant term $(z' + ax' + by')$ by c, it becomes

$$ax + by + z - c = 0, \qquad (6)$$

which is the equation required.

178. The intersections of a plane with the co-ordinate planes are called **the Traces of the Plane.**

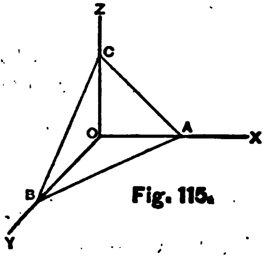

Fig. 115.

For every point in the plane xz, $y = 0$; if then we substitute $y = 0$ in (6) of Art. 177 and solve for x, we have

$$x = -\frac{1}{a}z + \frac{c}{a}; \qquad (1)$$

which is the *equation of the trace* AC on the plane xz.

Similarly

$$y = -\frac{1}{b}z + \frac{c}{b}, \qquad (2)$$

is *the equation of the trace* BC on the plane yz.

Also
$$y = -\frac{a}{b}x + c; \tag{3}$$
is *the equation of the trace* AB on the plane xy.

Comparing the coefficients of z in the equations of the traces (1) and (2), with the coefficients of z in (1) of Art. 177, we see that the traces on the planes xz and yz are perpendicular to the projections of the director on the same planes; and the same may be shown for the third trace AB and the projection on the plane xy. Hence, if *a line in space is perpendicular to a plane, its projections are perpendicular to the traces of the plane.*

Cor.—If $c = 0$, (6) of Art. 177 becomes $ax + by + z = 0$, which is satisfied by $x = 0$, $y = 0$, $z = 0$, or the plane passes through the origin.

179. *Every equation of the first degree between three variables is the equation of a plane.*

The general equation of the first degree between three variables is of the form
$$Ax + By + Cz + D = 0. \tag{1}$$
Dividing by C, we have
$$\frac{A}{C}x + \frac{B}{C}y + z + \frac{D}{C} = 0, \tag{2}$$
an equation of the same nature and form as (6) of Art. 177, and therefore is the equation of a plane. Hence, *every equation of the first degree between three variables is the equation of a plane.*

Sch.—Comparing (2) of this Article with (6) of Art. 177, we see that $\frac{A}{C}$ and $\frac{B}{C}$ *are the tangents of the angles which the projections of the director on the planes* xz and yz make with the axis of z. (Art. 172.)

180. *To find the equation of a plane in terms of its intercepts on the co-ordinate axes.*

Let the intercepts of the plane

$$Ax + By + Cz + D = 0, \tag{1}$$

on the axes of x, y, and z, be a, b, and c, respectively.

Making y and z both $= 0$, and therefore $x = a$, (1) becomes

$$Aa + D = 0; \quad \therefore \quad A = -\frac{D}{a}.$$

Similarly making z and $x = 0$, and x and $y = 0$, we get

$$B = -\frac{D}{b}, \quad C = -\frac{D}{c}.$$

Substituting these values of A, B, and C, in (1), dividing by $-D$ and transposing, it becomes

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1; \tag{2}$$

which is *the equation required.*

This form is known as the *symmetrical form of the equation of a plane.* (See Art. 22, II.)

Rem.—Of course a, b, c, in (2), do not mean the same as a, b, c, in (6) of Art. 177.

181. *To find the equation of a plane in terms of the perpendicular on it, from the origin and the direction-cosines of this perpendicular.*

Let p be the perpendicular from the origin to the plane, and α, β, γ, its direction-angles (Art. 170); then since each intercept is equal to this perpendicular divided by the corresponding *direction-cosine,* we have

$$a = \frac{p}{\cos\alpha}; \quad b = \frac{p}{\cos\beta}; \quad c = \frac{p}{\cos\gamma};$$

which in (2) of Art. 180, give us

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p; \qquad (1)$$

which *is the equation required.*

This form is known as the *normal form of the equation of a plane.* (See Art. 22, III.)

SCH.—The general equation of the plane

$$Ax + By + Cz + D = 0, \qquad (2)$$

may be reduced to the form

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p, \qquad (3)$$

thus: Comparing (6) of Art. 177 with (2) of Art. 179, we have

$$a = \frac{A}{C}, \quad \text{and} \quad b = \frac{B}{C}.$$

Substituting these values of a and b in (5), (6), and (7) of Art. 175, we have

$$\cos \alpha = \frac{A}{\sqrt{A^2 + B^2 + C^2}};$$

$$\cos \beta = \frac{B}{\sqrt{A^2 + B^2 + C^2}};$$

$$\cos \gamma = \frac{C}{\sqrt{A^2 + B^2 + C^2}}.$$

Substituting these values in (3), we get

$$\frac{Ax + By + Cz + D}{\sqrt{A^2 + B^2 + C^2}} = 0; \qquad (4)$$

which is the required form. (See Art. 23, Cor.)

Comparing (4) with (3), we have for the perpendicular from the origin on the plane whose equation is (4),

$$p = \frac{-D}{\sqrt{A^2 + B^2 + C^2}}.$$

By giving to the square root the sign which will make this perpendicular positive, the resulting signs of the cosines will indicate whether the direction angles of the perpendicular are acute or obtuse.

182. *To find the length of the perpendicular from a given point* (x', y', z') *to a given plane,* $x \cos \alpha + y \cos \beta + z \cos \gamma = p$.

Pass a plane through the given point parallel to the given plane; its equation is (Art. 181)

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p'. \tag{1}$$

Since (x', y', z') is on this plane, it must satisfy its equation, giving us

$$x' \cos \alpha + y' \cos \beta + z' \cos \gamma = p'. \tag{2}$$

The length of the required perpendicular will be equal to the perpendicular distance between these parallel planes. The first member of (2) is the length of the perpendicular from the origin to the parallel plane through (x', y', z'), and p is the length of the perpendicular from the origin to the given plane; therefore,

$$x' \cos \alpha + y' \cos \beta + z' \cos \gamma - p,$$

is the perpendicular distance between the parallel planes, which is the length of the required perpendicular. Hence, the length of a perpendicular from a given point to a plane, is obtained by substituting the co-ordinates of that point in the normal equation of the plane.

SCH.—If the point (x', y', z') and the origin are on *opposite* sides of the given plane, this expression is *plus;* if they are on the *same* side, the expression is *minus;* therefore, the length of the perpendicular is

$$\pm (x' \cos \alpha + y' \cos \beta + z' \cos \gamma - p),$$

according as the point and the origin lie on *opposite* sides, or the *same* side, of the plane.

COR.—If the equation of the plane were given in the general form $Ax + By + Cz + D = 0$, we have only to reduce it to the form $x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0$ (Art. 181, Sch.), and the length of the perpendicular on it from any point (x', y', z') is

$$\pm \frac{Ax' + By' + Cz' + D}{\sqrt{A^2 + B^2 + C^2}}.$$

It is easily seen that (x', y', z') lies on the *same* side of the plane as the origin, or on the *opposite* side according as $Ax' + By' + Cz' + D$ has the *same* sign as D, or the *opposite* sign. (Art. 24, Sch.)

183. *To find the angle included between two planes.*

Let $$Ax + By + Cz + D = 0, \tag{1}$$

and $$A'x + B'y + C'z + D' = 0, \tag{2}$$

be the equations of the two planes. The angle between two planes is the same as the angle between two lines drawn perpendicular to them. If the equations of these two perpendiculars are

$$x = az + \alpha, \qquad y = bz + \beta;$$

and $$x = a'z + \alpha', \qquad y = b'z + \beta';$$

the angle between them is given by equation (8) of Art. 175.

Comparing (6) of Art. 177 with (2) of Art. 179, we have

$$a = \frac{A}{C}; \qquad b = \frac{B}{C};$$

$$a' = \frac{A'}{C'}; \qquad b' = \frac{B'}{C'};$$

which in (8) of Art. 175, gives us

$$\cos v = \frac{AA' + BB' + CC'}{\sqrt{A^2 + B^2 + C^2}\sqrt{A'^2 + B'^2 + C'^2}}; \quad (3)$$

which determines the angle between the two planes.

Cor. 1.—If the planes are parallel to each other, $\cos v = 1$; hence, clearing (3) of fractions, squaring, transposing and uniting (see Art. 175, Cor. 1), it becomes

$$(AB' - A'B)^2 + (BC' - B'C)^2 + (CA' - C'A)^2 = 0;$$

an equation which can be satisfied only when each term is equal to 0, giving us

$$AB' = A'B; \quad BC' = B'C; \quad CA' = C'A;$$

or $$\frac{A}{B} = \frac{A'}{B'}; \quad \frac{B}{C} = \frac{B'}{C'}; \quad \frac{A}{C} = \frac{A'}{C'}.$$

That is, if the planes are parallel to each other, the coefficients A, B, C, are proportional to A', B', C'.

Cor. 2.—If the planes are perpendicular to each other, $\cos v = 0$, and (3) becomes

$$AA' + BB' + CC' = 0,$$

which is the equation of condition that makes two planes perpendicular to each other.

Cor. 3.—If we suppose the second plane to coincide with the co-ordinate plane xy, we have $z = 0$ in (2), and it becomes

$$A'x + B'y = 0,$$

(since D also $= 0$, Art. 178, Cor.) And since this is true for every value of x and y, we shall have

$$A' = 0, \qquad B' = 0.$$

Therefore, denoting the angle between the plane (1) and the co-ordinate plane xy by v', we have from (3)

$$\cos v' = \frac{C}{\sqrt{A^2 + B^2 + C^2}}.$$

Calling v'' and v''' the angles which the plane (1) makes with the planes zx and yz, we obtain

$$\cos v'' = \frac{B}{\sqrt{A^2 + B^2 + C^2}};$$

$$\cos v''' = \frac{A}{\sqrt{A^2 + B^2 + C^2}}.$$

SCH.—$x = 0$, $y = 0$, $z = 0$, are the equations of the co-ordinate planes yz, zx, xy, respectively.

184. *To find the angle between the plane* $Ax + By + Cz + D = 0$ *and the line* $x = az + \alpha$, $y = bz + \beta$.

The angle between the line and the plane is the complement of the angle between the line and the perpendicular on the plane.

Let the equations of the perpendicular be

$$x = a'z + \alpha'; \qquad y = b'z + \beta'. \tag{1}$$

(8) of Art. 175 gives the value of the cosine of the angle between the line and the perpendicular; therefore, the value of the *sine* of the required angle between the line and plane is

$$\frac{aa' + bb' + 1}{\sqrt{a^2 + b^2 + 1}\sqrt{a'^2 + b'^2 + 1}}. \tag{2}$$

Since the line (1) is perpendicular to the plane, we have

$$a' = \frac{A}{C}; \qquad b' = \frac{B}{C};$$

which in (2) gives us, calling the required angle v',

$$\sin v' = \frac{Aa + Bb + C}{\sqrt{a^2 + b^2 + 1}\sqrt{A^2 + B^2 + C^2}}. \tag{3}$$

Cor. 1.—If the line is parallel to the plane, we have

$$Aa + Bb + C = 0. \tag{4}$$

Cor. 2.—If the line is perpendicular to the plane, we have

$$\sin v' = 1;$$

hence, clearing (4) of fractions, squaring, transposing, and uniting, we have

$$(aC - A)^2 + (bC - B)^2 + (aB - bA)^2 = 0,$$

an equation which can be satisfied only when each term $= 0$, giving us

$$aC = A, \quad bC = B, \quad aB = bA;$$

in which we see that the third term follows directly from the other two. Hence,

$$a = \frac{A}{C}, \text{ and } b = \frac{B}{C},$$

are the conditions that the lines shall be perpendicular to the plane, which are the same as we get by comparing (6) of Art. 177 with (2) of Art. 179.

EXAMPLES.

1. Find the equations of the traces of the plane $z + 2x + 3y = 6$, on each of the co-ordinate planes, and also the intercepts on each of the co-ordinate axes.

Ans. $2x + 3y = 6$, $z + 2x = 6$, $z + 3y = 6$; $a = 3$, $b = 2$, $c = 6$.

2. Find the z of the point in the same plane, whose projection on xy is (3, 4). *Ans.* $z = -12$.

3. Find the equation of a line passing through the point (— 2, 3, 5) and perpendicular to the plane

$$2x + 8y - z - 4 = 0.$$

Ans. $x = -2z + 8; \quad y = -8z + 43$.

4. Find the equation of a plane that shall pass through the points (1, — 2, 2), (0, 4, — 5) and (— 2, 1, 0).

Ans. $9x + 19y + 15z - 1 = 0$.

5. Find the line of intersection of the two planes

$$3x + 8y - 10z + 6 = 0;$$
$$4x - 8y + z + 1 = 0.$$

Ans. $7x - 9z + 7 = 0; \quad 56y - 43z + 21 = 0$.

[The co-ordinates of the line of intersection of two planes will satisfy, at the same time, the equations of both planes; therefore, combining these two equations and eliminating y, we obtain the equation of the projection, on the plane xz, of the intersection of the two planes.

In the same manner, by eliminating x we find the equation of the projection of the intersection on the plane yz.

This method may also be applied for determining the intersection of any two surfaces whatever, or of any line with any surface.]

6. Find the line of intersection of the two planes

$$z + 2x - y = 3,$$
$$z + x + 2y = 5.$$

Ans. $x = -\frac{3}{5}z + \frac{11}{5}; \quad y = -\frac{1}{5}z + \frac{7}{5}$.

7. Find the angle between the planes

$$5x - 7y + 3z + 1 = 0;$$
$$2x + y - 3z = 0.$$

Ans. 79° 52′.

8. Find the distance from the point (2, — 3, 0) to the plane

$$z - 8x - 9y - 2 = 0. \quad \text{(Art. 182.)}$$

Ans. — .75.

9. Find the angle which the plane

$$5x - 7y + 3z + 1 = 0,$$

makes with each of the co-ordinate planes. (Art. 183, Cor. 3.)

Ans. 70° 46′ with xy; 140° 12′ with zx; 56° 43′ with yz.

10. In Fig. 111, calling OM, ON, and OR, a, b, and c, respectively, find the equation of the plane passing through A, B, C. *Ans.* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 2.$

[The co-ordinates of the three points must satisfy (1) of Art. 179, giving us three equations from which to find the values of A, B, C; substituting these values of A, B, C, in (1), and dividing by D, we have the equation required.]

11. In the same Fig. find the equations of the plane passing through P, M, N. *Ans.* $\frac{x}{a} + \frac{y}{b} - \frac{z}{c} = 1.$

12. Find the equation of the plane passing through P, O, M; also the equation of the plane passing through P, O, N.

Ans. $\frac{y}{b} = \frac{z}{c}$; and $\frac{x}{a} = \frac{z}{c}$.

13. Find the length of the perpendicular from the origin on the plane in Ex. 10.

Ans. $\frac{2abc}{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}$. (See Art. 182.)

14. Find the length of the perpendicular from R on the plane in Ex. 11.

Ans. $\frac{2abc}{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}$.

15. Find the angle between the planes in Ex. 12.

$$Ans.\quad \cos^{-1}\frac{3}{\sqrt{a^2+b^2+c^2}\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}.$$

16. Find the condition that the right line $x = az + \alpha$, $y = bz + \beta$, shall lie wholly in the plane

$$Ax + By + Cz + D = 0.$$

$$Ans.\ Aa + Bb + C = A\alpha + B\beta + D = 0.$$

17. Prove that the sum of the squares of the cosines of the three angles which a plane makes with the three co-ordinate planes is equal to unity; also prove that the cosine of the angle between two planes is equal to the sum of the products of the cosines of the angles which the planes make with the co-ordinate planes.

CHAPTER IV.

SURFACES OF REVOLUTION.

185. A **Surface of Revolution** is a surface that can be generated by revolving any line about a fixed right line. The revolving line is called the **Generator**, and the fixed line around which it revolves is called the **Axis.** A section of the surface made by a plane passing through the axis is called a **Meridian Section,** and the plane a **Meridian Plane.**

From the definition it follows that every point in the generator describes the circumference of a circle whose centre is in the axis; hence the surface may be generated by revolving any meridian section about the axis.

186. *To find the general equation of a surface of revolution.*

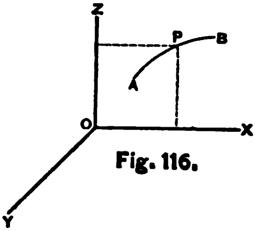

Fig. 116.

Let AB be any curve in the coordinate plane xz, and let it be revolved about the axis of z. Let (x, z) be any point P in the generator, and let (r, z) be the same point P in any position of the generator in its revolution about the axis of z, rz being any plane through the axis of z perpendicular to the plane xy. Then the equation of the generator may be written in the form

$$r = f(z), \tag{1}$$

which is read, *r equals a function of z;* that is, r equals an expression that involves z.

Now, from the nature of the surface, any point P of the generator must describe a circle whose centre is in the axis of z, and whose plane is perpendicular to this axis, that is, parallel to the plane xy; therefore we must have, in every position of the point P,

$$x^2 + y^2 = r^2. \tag{2}$$

Now, for any one value of r, we shall have one value of z from (1), and this value of z, with the corresponding values of x and y, are the co-ordinates of every point of one horizontal circle of the surface. Hence, if we suppose r to have every value that it can have and satisfy (1), the corresponding values of z with those of x and y, will be the co-ordinates of every point of every horizontal circle, that is, of every point of the surface. Therefore, combining (1) and (2), we have

$$x^2 + y^2 = \overline{f(z)}^2, \tag{3}$$

which is *the equation required.*

187. *To find the equation of a right circular cone.*

A **Right Cone** is a surface that may be generated by revolving a right line about another right line which it intersects. The point of intersection is called the **Vertex** of the cone. If the axis is perpendicular to the base, the cone is called a **Right Circular Cone**; the different positions of the generator are called **Elements of the Cone.**

Let the axis of the cone be the axis of z, and its base the plane xy. Then the co-ordinates of the vertex are $x' = 0$, $y' = 0$, $z' = c$. By Art. 25, the equation of the generator in the plane xz is

$$x = a(z - c).$$

Hence, Art. 186,

$$r = f(z) = a(z - c),$$

which in (3) of Art. 186 gives

$$x^2 + y^2 = a^2(z - c)^2, \tag{1}$$

where a is the tangent of the angle which the generator makes with the *axis* of the cone. If we let $\phi =$ the inclination of the generator to the *base* of the cone, we have $a = \cot \phi$, which in (1) gives

$$(x^2 + y^2) \tan^2 \phi = (z - c)^2, \tag{2}$$

which is *the equation required.*

Cor.—If the vertex be placed at the origin of co-ordinates, $c = 0$, and (1) becomes

$$x^2 + y^2 = a^2z^2. \tag{3}$$

188. *To find the equation of the right circular cylinder.*

A **Right Circular Cylinder** is a surface that may be generated by revolving a right line about another right line parallel to itself.

Let the axis of the cylinder be the axis of z, and its base the plane xy, and let a be the distance of the generator from the axis. The equation of the generator in the plane xz is (Art. 25)

$$x = a.$$

Hence, Art. 186, $r = f(z) = a$,

which in (3) of Art. 186 gives

$$x^2 + y^2 = a^2,$$

which is *the required equation.*

189. *To find the equation of the sphere.*

A **Sphere** is a surface that may be generated by revolving a circle about one of its diameters.

Let the plane of the generating circle begin in the plane xz, and let the axis of revolution be the axis of z. The equation of the generator in the plane xz is

$$x^2 + z^2 = R^2;$$

and in any other position of its revolution, the equation is

$$r^2 + z^2 = R^2 \text{ (Art. 186)},$$

or

$$r^2 = R^2 - z^2,$$

which in (3) of Art. 186 gives

$$x^2 + y^2 = R^2 - z^2,$$

or

$$x^2 + y^2 + z^2 = R^2, \tag{1}$$

which is *the equation required.*

190. *To find the equation of a paraboloid of revolution.*

A **Paraboloid of Revolution** is a surface that may be generated by revolving a parabola about its axis.

Let the plane of the generating parabola begin in the plane xz, and let the axis of the parabola be the axis of z. The equation of the generator in the plane xz is

$$x^2 = 2pz,$$

and in any other position of its revolution the equation is

$$r^2 = 2pz,$$

which in (3) of Art. 186 gives

$$x^2 + y^2 = 2pz, \tag{1}$$

which is *the equation required.*

COR.—Make $z =$ a constant $= q$, and (1) becomes

$$x^2 + y^2 = 2pq,$$

which is the equation of a circle whose radius is $\sqrt{2pq}$. Therefore, all sections of the paraboloid of revolution parallel to the plane xy are circles, real or imaginary, according as p and q have like or unlike signs.

If $x = m$, (1) becomes

$$y^2 = 2pz - m^2; \tag{2}$$

and if $y = n$, (1) becomes

$$x^2 = 2pz - n^2, \tag{3}$$

which are equations of parabolas. Therefore, all sections parallel to the planes yz and xz are parabolas.

191. An **Ellipsoid of Revolution** is a surface that may be generated by revolving an ellipse about one of its axes; if the ellipse revolves about its *major* axis, the surface generated is called a **Prolate Spheroid**; if it revolves about its *minor* axis, the surface is called an **Oblate Spheroid.**

192. *To find the equation of a prolate spheroid.*[*]

Let the plane of the generating ellipse begin in the plane xz, and let the *major* axis be the axis of z. The equation of the generator in the plane xz is

$$a^2x^2 + b^2z^2 = a^2b^2;$$

---

[*] The equation of *any* ellipsoid may be found as follows:

Let
$$Ax^2 + By^2 + Cz^2 = D \tag{1}$$
be an equation of the second degree between three variables, containing only the squares of the variables and the absolute term, and let A, B, C, and D all be positive.

If $x = m$, $y = n$, and $z = q$ successively, (1) becomes in succession,

$$By^2 + Cz^2 = D - Am^2, \tag{2}$$

$$Ax^2 + Cz^2 = D - Bn^2, \tag{3}$$

$$Ax^2 + By^2 = D - Cq^2, \tag{4}$$

which are equations of ellipses, real or imaginary, according as the second members are positive or negative; that is, (2) is a real ellipse if

$$m < \pm\sqrt{\frac{D}{A}},$$

(3) is a real ellipse if
$$n < \pm\sqrt{\frac{D}{B}},$$

and (4) is a real ellipse if
$$q < \pm\sqrt{\frac{D}{C}}.$$

and in any other position of its revolution the equation is

$$a^2r^2 + b^2z^2 = a^2b^2,$$

or

$$r^2 = \frac{a^2b^2 - b^2z^2}{a^2};$$

$$m = \pm\sqrt{\frac{D}{A}},$$

$$n = \pm\sqrt{\frac{D}{B}},$$

$$q = \pm\sqrt{\frac{D}{C}},$$

(2), (3), and (4) are equations of points. Therefore all sections of the figure represented by (1), parallel to the planes xy, yz, zx, are ellipses, and the figure is limited by six parallel planes,

two of them at the distances $+\sqrt{\frac{D}{A}}$ and $-\sqrt{\frac{D}{A}}$ from the plane yz,

two at the distances $+\sqrt{\frac{D}{B}}$ and $-\sqrt{\frac{D}{B}}$ from the plane xz,

and two at the distances $+\sqrt{\frac{D}{C}}$ and $-\sqrt{\frac{D}{C}}$ from the plane xy,

and hence the surface is called an *ellipsoid*.

Let the intercepts of the ellipsoid on the axes of x, y, and z be a, b, c, respectively. Making z and $y = 0$, and therefore $x = a$, (1) becomes

$$Aa^2 = D; \qquad \therefore\ A = \frac{D}{a^2}.$$

Similarly, by making z and $x = 0$, and x and $y = 0$, we get

$$B = \frac{D}{b^2}, \qquad C = \frac{D}{c^2}.$$

Substituting these values in (1) and dividing by D, we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{5}$$

which is *the equation of the ellipsoid referred to its centre and axes.*

If $c = b$, (5) becomes $a^2(y^2+z^2)+b^2x^2 = a^2b^2$, which is an *ellipsoid of revolution* round the axis of x.

If $c = a$, (5) becomes $b^2(x^2+z^2)+a^2y^2 = a^2b^2$, which is an *ellipsoid of revolution* round the axis of y.

If $b = a$, (5) becomes $c^2(x^2+y^2)+a^2z^2 = a^2c^2$, which is an *ellipsoid of revolution* round the axis of z.

If $a = b = c = r$, (5) becomes $x^2+y^2+z^2 = r^2$, which is a *sphere*.

which in (3) of Art. 186 gives

$$x^2 + y^2 = \frac{a^2b^2 - b^2z^2}{a^2},$$

or

$$a^2(x^2 + y^2) + b^2z^2 = a^2b^2, \tag{1}$$

which is *the equation required.*

Cor. 1.—If $b = a$, (1) becomes $x^2 + y^2 + z^2 = a^2$, which is the equation of a sphere whose radius is a.

Cor. 2.—If $z = q$, (1) becomes

$$x^2 + y^2 = \frac{b^2}{a^2}(a^2 - q^2), \tag{2}$$

which is the equation of a circle whose radius $= \frac{b}{a}\sqrt{a^2 - q^2}$. Therefore all sections of the prolate spheroid parallel to the plane xy are circles, real or imaginary, according as $q^2 <$ or $> a^2$; if $q^2 = a^2$, the circle is a point. Hence, the surface is limited by two planes parallel to the plane xy, and at distances from it $= +a$ and $-a$.

If $x = m$, (1) becomes

$$a^2y^2 + b^2z^2 = a^2(b^2 - m^2); \tag{3}$$

and if $y = n$, (1) becomes

$$a^2x^2 + b^2z^2 = a^2(b^2 - n^2), \tag{4}$$

which are equations of ellipses. The ellipse whose equation is (3) is real or imaginary, according as $m^2 <$ or $> b^2$; if $m^2 = b^2$, the ellipse is a point.

Similarly, (4) is real or imaginary, or a point, according as $n^2 < b^2$, $> b^2$, or $= b^2$. Therefore, all sections of the prolate spheroid parallel to the planes yz and zx are ellipses; and the surface is limited by four parallel planes, two of them at the distances $+b$ and $-b$ from the plane yz, and two of them at the same distances from the plane zx.

193. *To find the equation of an oblate spheroid.*

Let the plane of the generating ellipse begin in the plane xz, and let the *minor* axis be the axis of z. The equation of the generator in the plane xz is

$$a^2z^2 + b^2x^2 = a^2b^2;$$

and in any other position of its revolution, the equation is

$$a^2z^2 + b^2r^2 = a^2b^2,$$

or $$r^2 = \frac{a^2b^2 - a^2z^2}{b^2},$$

which in (3) of Art. 186 gives

$$x^2 + y^2 = \frac{a^2b^2 - a^2z^2}{b^2},$$

or $$b^2(x^2 + y^2) + a^2z^2 = a^2b^2, \qquad (1)$$

which is *the equation required.*

SCH.—This equation is the same as (1) of Art. 192, except that a and b change places. Hence, by changing a to b and b to a in Cors. 1 and 2, Art. 192, the conclusions of that Art. may be applied to the oblate spheroid.

194. An **Hyperboloid of Revolution** is a surface that may be generated by revolving an hyperbola about one of its axes. If the hyperbola revolves about its *transverse* axis, the surface generated is called an **Hyperboloid of Revolution of Two Nappes**; each branch of the hyperbola generates a separate nappe, or branch of the surface. If the hyperbola revolves about its *conjugate* axis, the surface generated is called an **Hyperboloid of Revolution of One Nappe.**

195. *To find the equation of an hyperboloid of revolution of two nappes.*

Changing b^2 into $-b^2$ in (1) of Art. 192, it becomes (see Art. 102, Cor. 5)

$$a^2(x^2+y^2)-b^2z^2=-a^2b^2, \qquad (1)$$

which is *the required equation.*

COR.—If $z=q$, (1) becomes

$$x^2+y^2=\frac{b^2}{a^2}(q^2-a^2), \qquad (2)$$

which is the equation of a circle whose radius $=\frac{b}{a}\sqrt{q^2-a^2}$. Therefore, all sections of the hyperboloid of revolution of two nappes, parallel to the plane xy, are circles, real or imaginary, according as $q^2 >$ or $< a^2$; if $q^2=a^2$, the circle is a point. Hence, the surface is limited by two planes parallel to the plane xy, and at distances from it $=a$ and $-a$, no part of the surface being *between* the limiting planes.

If $x=m$, (1) becomes

$$a^2y^2-b^2z^2=-a^2(b^2+m^2), \qquad (3)$$

and if $y=n$, (1) becomes

$$a^2x^2-b^2z^2=-a^2(b^2+n^2), \qquad (4)$$

which are equations of hyperbolas, whose transverse axes are all parallel to the axis of z. Therefore, all sections of the hyperboloid of revolution of two nappes, parallel to the planes yz and xz, are hyperbolas.

196. *To find the equation of an hyperboloid of revolution of one nappe.*

Changing b^2 into $-b^2$ in (1) of Art. 193, it becomes

$$b^2(x^2+y^2)-a^2z^2=a^2b^2, \qquad (1)$$

which is *the required equation* (Art. 102, Cor. 5).

COR.—If $z=q$, (1) becomes

$$x^2+y^2=\frac{a^2}{b^2}(b^2+q^2), \qquad (2)$$

which is the equation of a circle whose radius $= \frac{a}{b}\sqrt{b^2+q^2}$, and this circle is real for *every value of q*. Therefore, all sections of the hyperboloid of revolution of one nappe, parallel to the plane xy are real circles, and the surface has no limit in the direction of the axis of z. The smallest circle is that obtained by making $z = 0$ in (1), giving us

$$x^2 + y^2 = a^2, \tag{3}$$

which is called the **Circle of the Gorge.**

If $x = m$, (1) becomes

$$b^2y^2 - a^2z^2 = b^2(a^2 - m^2); \tag{4}$$

and if $y = n$, (1) becomes

$$b^2x^2 - a^2z^2 = b^2(a^2 - n^2), \tag{5}$$

which are the equations of hyperbolas whose transverse axes are all parallel to the axis of z, if m^2 and n^2 are $> a^2$; but if $m^2 < a^2$, the hyperbola represented by (4) has its axis parallel to the axis of y; and if $n^2 < a^2$, the one represented by (5) has its axis parallel to the axis of x. Therefore, all sections parallel to the planes yz and zx are hyperbolas. If $m^2 = a^2$, (4) becomes

$$y = \pm\frac{a}{b}z; \tag{6}$$

and if $n^2 = a^2$, (5) becomes

$$x = \pm\frac{a}{b}z. \tag{7}$$

Each of the equations (6) and (7) represents two right lines intersecting at the origin. The lines in (6) are the asymptotes of the hyperbolas parallel to the plane yz, and the lines in (7) are the asymptotes of the hyperbolas parallel to the plane xz (Art. 114).

Sch.—In finding the equations of the ellipsoids and hyperboloids of revolution, we made in each case the axis of

z the axis of revolution. If we wish to find the equations of these surfaces when the revolution is around some other axis, as, for example, the axis of x, we have only to interchange x and z in the above equations, and the resulting equation will be the required one.

SECTIONS OF A CONE.

197. We shall now show that if a right circular cone be cut by a plane, the curve of intersection will be one of the *conic sections* (see Art. 51, Rem.).

To find the equation of the intersection of a right circular cone and a plane.

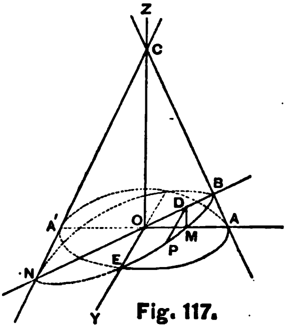

Fig. 117.

Let the axis of the cone be the axis of z, and the base of the cone the plane xy; and denote the angle OAC by ϕ; then the equation of the cone [Art. 187, (2)] is

$$(x^2+y^2)\tan^2\phi = (z-c)^2. \quad (1)$$

Pass the plane YOB through the axis of y, intersecting the cone in the curve BEN. Let $\theta =$ the angle AOB which the intersecting plane makes with the plane xy. Since the plane YOD is perpendicular to the plane xz, the lines OY and OD are perpendicular to each other; take OB and OY for the axes of x and y.

Let P be any point of the curve of intersection. Its coordinates referred to the old axes are

$$x = \text{OM}, \qquad y = \text{PD}, \qquad z = \text{DM};$$

and referred to the new axes OB and OY are

$$x' = \text{OD}, \qquad y' = \text{PD}.$$

In the right-angled triangle OMD, we have

$$\text{OM} = \text{OD} \cos\theta, \quad \text{or} \quad x = x' \cos\theta;$$

and $$\text{DM} = \text{OD} \sin\theta, \quad \text{or} \quad z = x' \sin\theta;$$

and $$y = y'.$$

Substituting these values of x, y, and z in (1), we have

$$(x'^2 \cos^2\theta + y'^2) \tan^2\phi = (x' \sin\theta - c)^2.$$

Performing the operations indicated, transposing, and omitting accents, we have

$$x^2 (\cos^2\theta \tan^2\phi - \sin^2\theta) + y^2 \tan^2\phi + 2cx \sin\theta - c^2 = 0. \quad (2)$$

$$\sin^2\theta = \cos^2\theta \tan^2\theta;$$

which substituted in (2) gives

$$x^2 (\tan^2\phi - \tan^2\theta) \cos^2\theta + y^2 \tan^2\phi + 2cx \sin\theta = c^2, \quad (3)$$

which is *the equation of the intersection of a right circular cone and a plane referred to two rectangular axes in the plane.*

By giving to c every value from 0 to CO, the axis of the cone, and by giving to θ every value from 0 to 90°, (3) will represent in succession every section that can be cut from a given right circular cone by a plane.

COR.—Since (3) is of the second degree, every section of a right circular cone will belong to one of the three classes (Art. 146, Sch.), which are characterized as follows:

$b^2 - 4ac = 0$, the parabola,

$b^2 - 4ac < 0$, the ellipse,

$b^2 - 4ac > 0$, the hyperbola.

Comparing (3) with the *general equation* of the second degree (Art. 141), we have

$$a = (\tan^2\phi - \tan^2\theta) \cos^2\theta,$$

$$b = 0,$$

$$c = \tan^2\phi.$$

I. Let $\theta = \phi$; we then have $a = 0$, or $b^2 - 4ac = 0$; hence the conic section is a *parabola.* In this case *the cutting plane is parallel to the side of the cone.*

1. If $c = 0$, the cutting plane passes through the vertex, and (3) becomes $y^2 \tan^2 \phi = 0$, or $y = 0$, which is the equation of the axis of x, showing that the limiting case of the parabola is a right line (Art. 146, Cor.).

2. If we suppose $\theta = \phi = 90°$ and $c = \infty$, (3) becomes $y^2 =$ constant; showing that when the vertex of the cone recedes to infinity, the parabola breaks up into two parallels. Therefore, *one right line* and *two parallel right lines* are the limiting cases of the parabola. (Art. 146, Cor.)

II. Let $\theta < \phi$; we then have $a > 0$, or $b^2 - 4ac < 0$; hence the conic section is *an ellipse.* In this case the cutting plane makes a less angle with the base than that made by the side of the cone.

1. If $\theta = 0$, the cutting plane is parallel to the base of the cone, and (3) becomes

$$x^2 + y^2 = c^2 \cot^2 \phi,$$

which is the equation of a *circle.*

2. If $c = 0$, the cutting plane passes through the vertex, and (3) becomes

$$x^2 (\tan^2 \phi - \tan^2 \theta) \cos^2 \theta + y^2 \tan^2 \phi = 0.$$

Each term, in the first member of this equation, being a square, is essentially positive; and hence the equation is satisfied only for the values $x = 0$, $y = 0$, which are the equations of the *origin.*

Therefore, a *circle* and *point* are the limiting cases of the *ellipse.* (Art. 143, Cor. 2.)

III. Let $\theta > \phi$; we then have $a < 0$, or $b^2 - 4ac > 0$; hence, the conic section is an *hyperbola.* In this case the cutting plane makes a greater angle with the base than that made by the side of the cone.

1. If θ has such a value as that,

$$(\tan^2 \phi - \tan^2 \theta) \cos^2 \theta + \tan^2 \phi = 0,$$

we shall have

$$a + c = 0;$$

and therefore (Art. 143, Cor. 3), the section is an *equilateral hyperbola.*

2. If $c = 0$, the cutting plane passes through the vertex, and (3) becomes

$$x^2 (\tan^2 \theta - \tan^2 \phi) \cos^2 \theta - y^2 \tan^2 \phi = 0;$$

or $$y = \frac{x \cos \theta \sqrt{\tan^2 \theta - \tan^2 \phi}}{\tan \phi};$$

which is the equation of two right lines intersecting at the origin.

Therefore, the *equilateral hyperbola and two intersecting right lines*, are the limiting cases of the hyperbola. (Art. 143, Cor. 3.)

Thus, we have shown that if a right circular cone be cut by a plane, the curve of intersection is a *parabola*, an *ellipse*, an *hyperbola*, a *circle*, *one right line*, *two right lines*, or a *point*. (See Art. 51, Remark.) The *imaginary* varieties cannot be obtained by any geometric process, since they have no *geometric* meaning. (See Art. 45.)

198. *To find the equation of a plane tangent to an ellipsoid at a given point.*

A plane is tangent to a surface when it has at least one point in common with it, through which if any number of planes be drawn, the sections made in the plane will be tangent to the sections made in the surface.

Let (x', y', z') be any given point on the ellipsoid,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{1}$$

Since (x', y', z') is on the ellipsoid, it will satisfy (1); therefore we have

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} = 1. \tag{2}$$

Subtracting (2) from (1) and factoring, we have

$$\frac{1}{a^2}(x-x')(x+x') + \frac{1}{b^2}(y-y')(y+y') + \frac{1}{c^2}(z-z')(z+z') = 0; \tag{3}$$

which is the equation of the ellipsoid with the condition introduced that the point (x', y', z') shall be on the surface.

The equations of a right line passing through (x', y', z') are (Art. 173)

$$x - x' = a'(z - z'); \qquad y - y' = b'(z - z'). \tag{4}$$

Substituting these values of $x - x'$ and $y - y'$ in (3), it becomes

$$\frac{a'}{a^2}(z-z')(x+x') + \frac{b'}{b^2}(z-z')(y+y') + \frac{1}{c^2}(z-z')(z+z') = 0; \tag{5}$$

which shows the relation between the co-ordinates $x, y, z,$ of every point common to the surface (3) and the line (4). (See Art. 184, Ex. 5.) Because (5) is of the second degree, there are *two* points common to the line and surface. Solving (5) we get

$$z - z' = 0; \tag{6}$$

and $$\frac{a'}{a^2}(x + x') + \frac{b'}{b^2}(y + y') + \frac{1}{c^2}(z + z') = 0. \tag{7}$$

Combining (4) and (6), we get

$$x = x', \qquad y = y'; \qquad z = z';$$

which are the co-ordinates of the *given* point, while the x, y, z, in (7) are the co-ordinates of the *second* point common to the line and surface.

If we pass a plane through this line, it will cut from the surface a line which will contain both the given point and the second point. If the second point be moved along this line till it becomes consecutive with the given point, the right line will become tangent to the line cut from the surface, at the given point, and x, y, z, will become x', y', z', which in (7) give

$$\frac{a'}{a^2}x' + \frac{b'}{b^2}y' + \frac{z'}{c^2} = 0; \tag{8}$$

which is the equation of condition that makes (4) tangent to a line of the surface at the given point.

Substituting in (8) the values of a' and b' found in (4), and clearing of fractions, we have

$$\frac{x'}{a^2}(x - x') + \frac{y'}{b^2}(y - y') + \frac{z'}{c^2}(z - z'). \tag{9}$$

Now, for *one set* of values of a' and b' in (4) that will satisfy (8), x, y, z, in (9) are the co-ordinates of every point of the tangent to *one* line of the surface, at the given point. Therefore, for *every set* of values of a' and b' in (4) that will satisfy (8), x, y, z, in (9) are the co-ordinates of every point of the tangent to *every* line of the surface, at the given point; that is, they are the co-ordinates of the *plane* tangent to the surface at the given point. Hence, (9) is the equation of the tangent plane at the point (x', y', z').

Performing the operations indicated in (9) and transposing, we have

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = \frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2}.$$

Substituting from (2), we have

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1; \tag{10}$$

which is *the equation required.*

If $c = b$, (10) becomes

$$a^2(yy' + zz') + b^2xx' = a^2b^2;$$

which is *the tangent plane to the ellipsoid of revolution round the axis of x.*

If $c = a$, (10) becomes

$$b^2(xx' + zz') + a^2yy' = a^2b^2;$$

which is *the tangent plane to the ellipsoid of revolution round the axis of y.*

If $a = b = c = r$, (10) becomes

$$xx' + yy' + zz' = r^2;$$

which is *the tangent plane to the sphere.*

EXAMPLES.

1. Find the equation of the cone generated by revolving about the axis of z, the line whose equation in the plane xz is $4x = 3z + 2$ (Art. 187), and find the vertex of the cone.
Ans. $16x^2+16y^2-9z^2-12z = 4$; vertex is at $(0, 0, -\frac{2}{3})$.

2. Find the equation of the cone generated by revolving about the axis of z the line whose equation in the plane yz is $2y + z = 6$, and find the vertex of the cone.

Ans. $x^2 + y^2 - \frac{z^2}{4} + 3z = 9$; vertex at $(0, 0, 6)$.

3. Find the equation of a right circular cone, the equation of the base being $x^2 + y^2 = 9$, and the altitude being 5.
Ans. $25x^2 + 25y^2 - 9z^2 + 90z = 225$.

4. Find the equation of the paraboloid of revolution generated by revolving the parabola $2x^2 = 5z$, about the axis of z. (Art. 190.) *Ans.* $x^2 + y^2 = \frac{5}{2}z$.

5. Find the equation of the paraboloid of revolution generated by revolving the parabola $y^2 = -3x$, about the axis of x. (Art. 196, Sch.) *Ans.* $y^2 + z^2 + 3x = 0$.

[How is this paraboloid situated relative to the coordinate axes?]

6. Find the equations of the spheroids generated by the ellipse $4x^2 + z^2 = 4$. (Arts. 192, 193.)

$$\textit{Ans.}\begin{cases}\text{The prolate, } 4x^2 + 4y^2 + z^2 = 4.\\ \text{The oblate, } 4x^2 + y^2 + z^2 = 4.\end{cases}$$

7. Find the equations of the hyperboloids generated by the hyperbola $9z^2 - 4x^2 = -36$. (Arts. 195, 196.)

$$\textit{Ans.}\begin{cases}\text{Of two nappes, } 9y^2 + 9z^2 - 4x^2 + 36 = 0,\\ \text{Of one nappe, } 4x^2 + 4y^2 - 9z^2 - 36 = 0.\end{cases}$$

8. Find the equations of the spheroids generated by the ellipse $16y^2 + 9x^2 = 144$. (Art. 196, Sch.)

$$\textit{Ans.}\begin{cases}\text{The prolate, } 9x^2 + 16y^2 + 16z^2 = 144,\\ \text{The oblate, } 9x^2 + 16y^2 + 9z^2 = 144.\end{cases}$$

9. Find the equations of the hyperboloids generated by the hyperbola $9y^2 - x^2 = -9$. (Art. 196, Sch.)

$$\textit{Ans.}\begin{cases}\text{Of two nappes, } 9y^2 + 9z^2 - x^2 + 9 = 0,\\ \text{Of one nappe, } x^2 - 9y^2 + z^2 - 9 = 0.\end{cases}$$

10. Find the equation of the surface generated by revolving the hyperbola $2y^2 = x$ about the axis of y.

Ans. $x^2 - 4y^4 + z^2 = 0$.

11. Find the equations of the surfaces generated by revolving $y^2 = \dfrac{1}{x}$ and $y^3 = 2x^2$ about the axis of y.

$$\textit{Ans.}\begin{cases}x^2 - \dfrac{1}{y^4} + z^2 = 0,\\ x^2 - \frac{1}{2}y^3 + z^2 = 0.\end{cases}$$

12. Find the eccentricity of the ellipse formed by the intersection of the ellipsoid $2x^2 + 3y^2 + 4z^2 = 1$ and the plane $z = \frac{1}{4}$. (Art. 184, Ex. 5.) *Ans.* $e = \sqrt{\frac{1}{3}}$.

13. Intersect the cone in Ex. 3 by a plane passing through the axis of y and making an angle of 45° with the base, and find the equation of the curve of intersection in its own plane (Art. 197); also find the axes of the curve of intersection (Art. 143, Cor. 2), and its eccentricity.

$$Ans. \begin{cases} 8x^2 + 25y^2 + 45\sqrt{2}x = 225; \\ \text{axes are } \dfrac{75}{8\sqrt{2}} \text{ and } \dfrac{15}{4}; \\ \text{eccentricity} = \frac{1}{5}\sqrt{17}. \end{cases}$$

14. Find the equations of the intersection of the sphere

$$x^2 + y^2 + z^2 = 16$$

and the ellipsoid of revolution

$$25(x^2 + y^2) + 9z^2 = 225,$$

their centres being coincident.

Ans. $x^2 + y^2 = \frac{81}{16}$ and $z = \pm \frac{5}{4}\sqrt{7}$.

15. Find the length of the perpendicular from the origin to the plane tangent to the ellipsoid at the point (x', y', z').

$$Ans.\ p = \frac{1}{\sqrt{\dfrac{x'^2}{a^4} + \dfrac{y'^2}{b^4} + \dfrac{z'^2}{c^4}}}.$$

[This easily follows from Art. 182, Cor.]

16. Find the length of the perpendicular from the origin to the plane tangent to the ellipsoid in Ex. 12 at the point $(\frac{1}{4}, \frac{1}{4}, \frac{1}{8}\sqrt{11})$. *Ans.* $p = \frac{1}{4}\sqrt{57}$.

17. Find a point on an ellipsoid such that the tangent plane cuts off equal intercepts from the axes. Also find a

point such that the intercepts cut off by the tangent plane are proportional to the axes.

$$Ans. \begin{cases} \text{First, } \dfrac{x'}{a^2} = \dfrac{y'}{b^2} = \dfrac{z'}{c^2} = \dfrac{1}{\sqrt{a^2+b^2+c^2}}; \\ \text{Second, } \dfrac{x'}{a} = \dfrac{y'}{b} = \dfrac{z'}{c} = \dfrac{1}{\sqrt{3}}. \end{cases}$$

18. Find the equation of the normal line to a tangent plane to the ellipsoid at the point of contact. (See Art. 184, Cor. 2.)

$$Ans. \begin{cases} \dfrac{a^2}{x'}(x - x') = \dfrac{c^2}{z'}(z - z'). \\ \dfrac{b^2}{y'}(y - y') = \dfrac{c^2}{z'}(z - z'). \end{cases}$$

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